Benchmark problem 1

The third international workshop on long-wave runup models
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Theory employed

The models

- **Eulerian FDM for standard Boussinesq equations**
  Nonlinear code without particular run-up features
  Used only in linear mode, with and without dispersion terms
- **Lagrangian FDM for Boussinesq type equations**
  Method particularly designed for runup
  Fully nonlinear, but weakly dispersive. Employed in hydrostatic and dispersive mode
- **Boundary Integral Method (BIM)**
  Full potential theory
  Used for comparison

Definitions

- h equilibrium depth, $h' = \partial h(x,t)/\partial x$ etc.
- η surface elevation
- $H = h + \eta$ total depth
- $\overline{u}$ depth-averaged horizontal velocity

Averaged material derivate

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}$$

The fully nonlinear Boussinesq equations

$$\frac{\mathrm{D}H}{\mathrm{D}t} = -H\frac{\partial \overline{u}}{\partial x},$$

$$\left(1 - \frac{1}{2}Hh'' - h'\frac{\partial \eta}{\partial x}\right)\frac{\mathrm{D}\overline{u}}{\mathrm{D}t} =$$

$$-g\frac{\partial \eta}{\partial x} - \frac{1}{3}H\frac{\partial}{\partial x}\left(\frac{\mathrm{D}^2 H}{\mathrm{D}t^2}\right) - \frac{2}{3}\frac{\partial H}{\partial x}\frac{\mathrm{D}^2 H}{\mathrm{D}t^2}$$

$$+\frac{h'}{H}\left(\frac{\mathrm{D}H}{\mathrm{D}t}\right)^2 - h''\overline{u}\frac{\mathrm{D}H}{\mathrm{D}t} + \left(\frac{\partial \eta}{\partial x}h'' + \frac{1}{2}Hh'''\right)u^2 + S_1 + S$$

Red: “Important” dispersion terms

Green: fully nonlinear version, $S_1 + S_2$: source terms.

Long waves and Lagrangian coordinates

Simple flow structure $\Rightarrow$ material fluid columns remain (nearly) vertical

Lagrangian coordinate a: $\frac{\mathrm{D}a}{\mathrm{D}t} = 0, \quad a(x,0) = x$

Transformation $\frac{\mathrm{D}}{\mathrm{D}t} \to \frac{\partial}{\partial t}$, $\frac{\partial}{\partial x} \to \frac{H}{H_0}\frac{\partial}{\partial a}$ etc., $H_0 = H(a,0)$

$$H\frac{\partial x}{\partial a} = H_0,$$

$$(1 - () - ())\,\frac{\partial \overline{u}}{\partial t} = -g\frac{H}{H_0}\frac{\partial H}{\partial a} + g\frac{\partial h}{\partial x} - () + () + S_1 + S_2,$$

$$\frac{\partial x}{\partial t} = \overline{u},$$

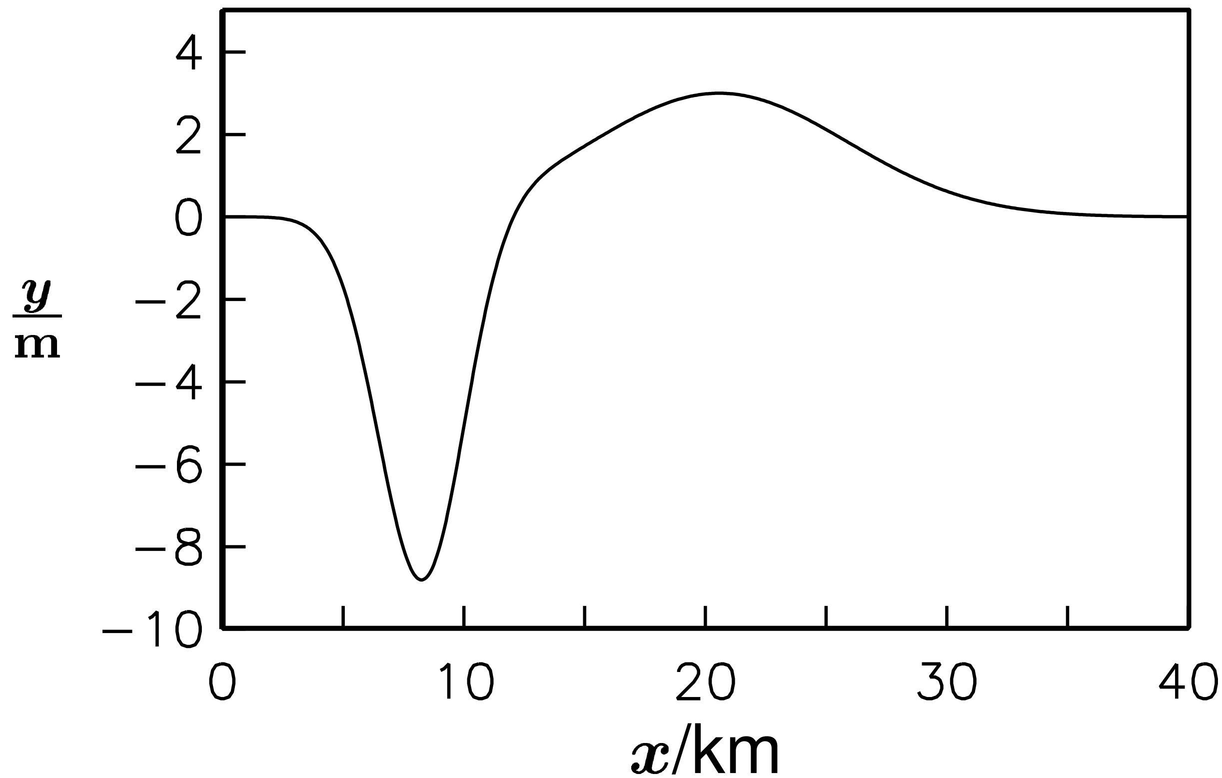
Shoreline: $H = 0$ at fixed a

The Lagrangian Boussinesq model

- Second order FDM model. Staggered grid in space and time.
- H-node at shoreline position (fixed a).
  Condition $H \equiv h + \eta = 0$ implemented directly.
- Hydrostatic (NLSW) version is explicit.
- Dispersive version implicit – iteration on nonlinearity (may be avoided)
- Wave paddle is also simple to implement.
- No smoothing or filtering for non-breaking waves.
- Hydrostatic (NLSW) extensions to non-planar waves exist

Results

Initial condition, “N-wave”



Shape akin to slide generated tsunami, but zero initial velocities. Resolution 50m.

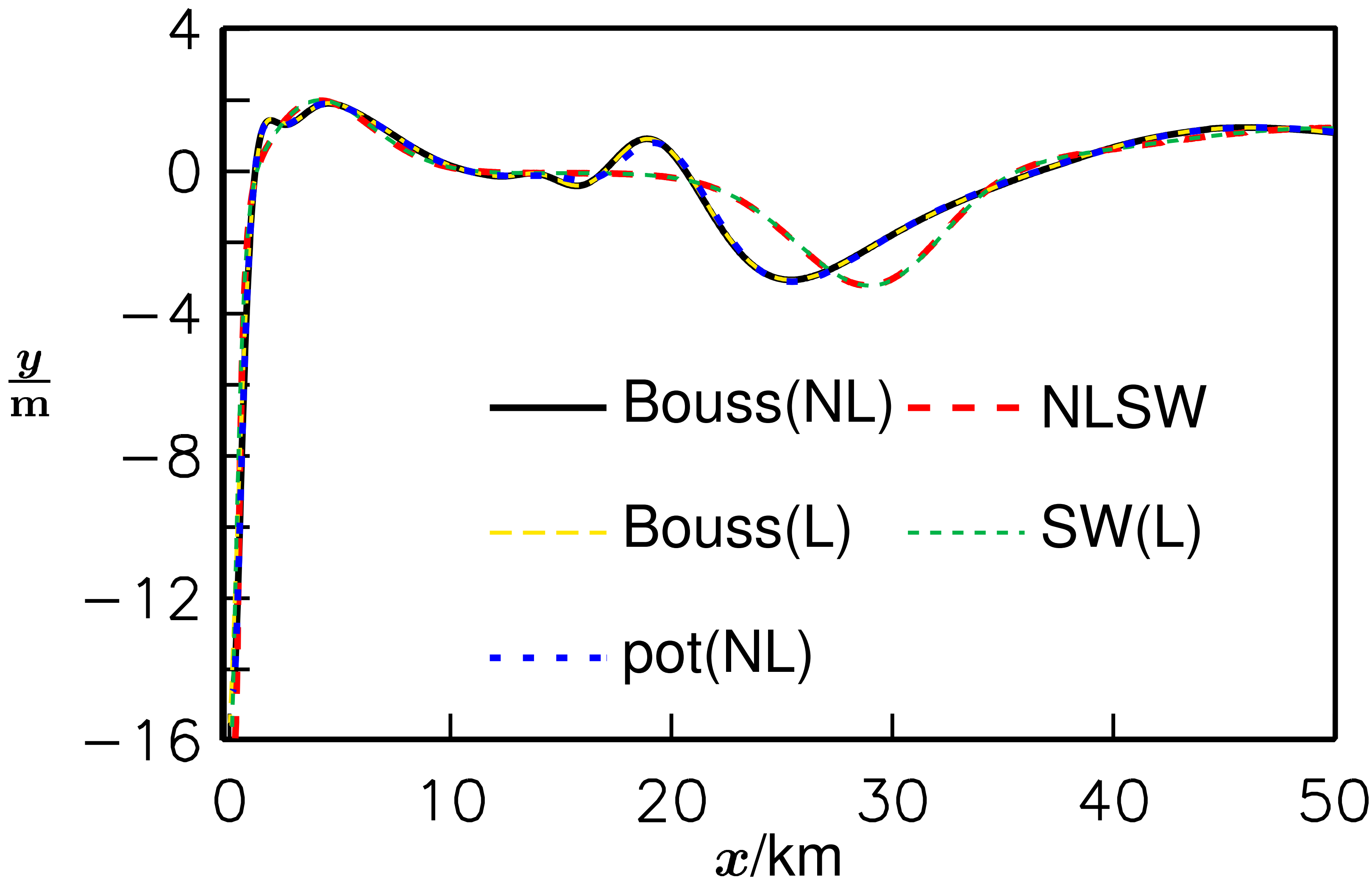
Bottom: linear, steep, 1 in 10 slope

Nomenclature

- ref.: (Semi-)analytic solutions from reference papers
- Bouss(NL) : Nonlinear Boussinesq solution (Lagrangian)
- NLSW: Nonlinear shallow water solution (Lagrangian)
- Bouss(L): Linear Boussinesq solution (Eulerian)
- SW(L): Linear shallow water solution (Eulerian)
- pot(NL): Full potential theory (boundary integral method)

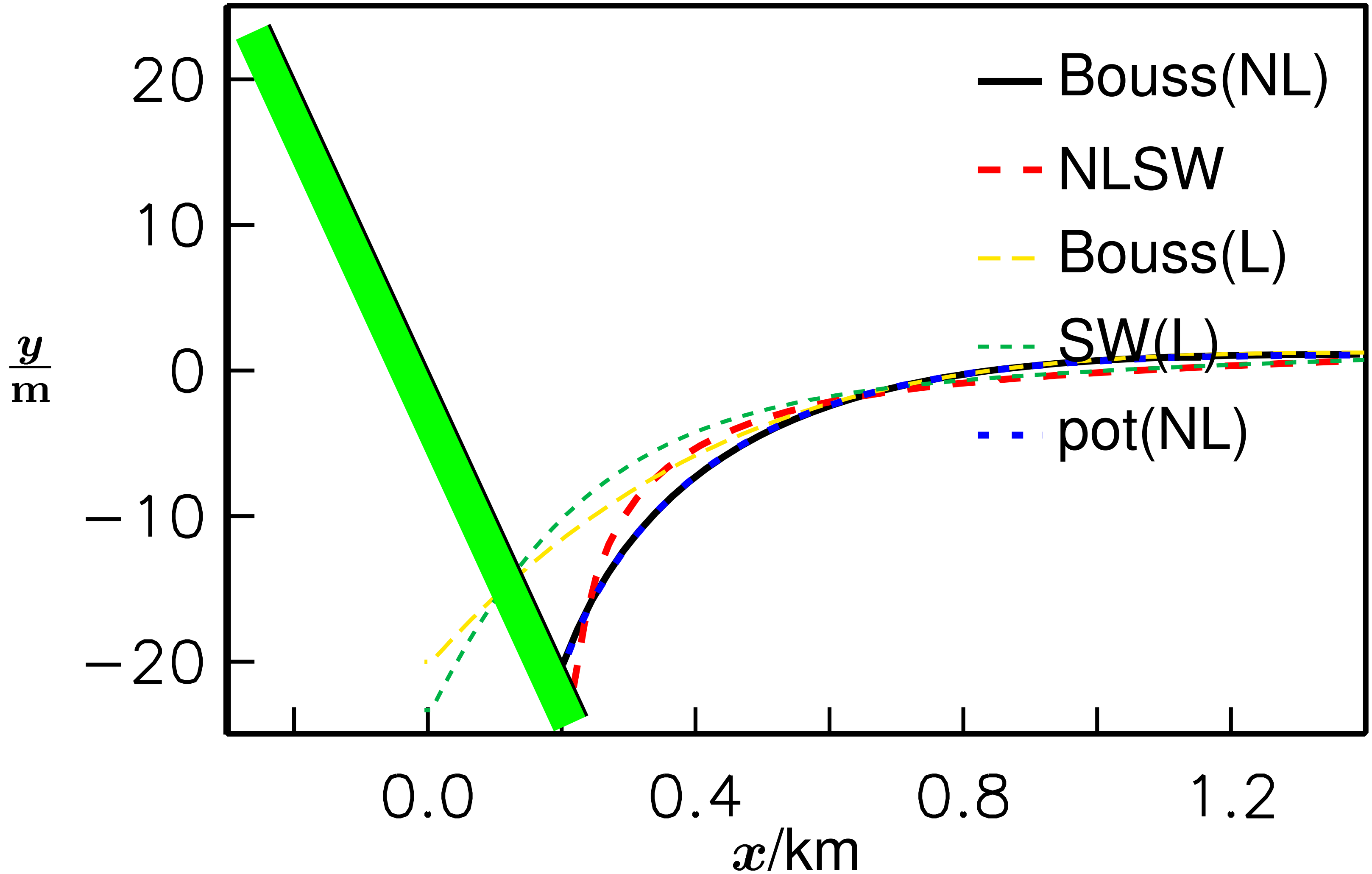
Color codes are preferably kept consistent.

Global; $t = 160.0$ sec



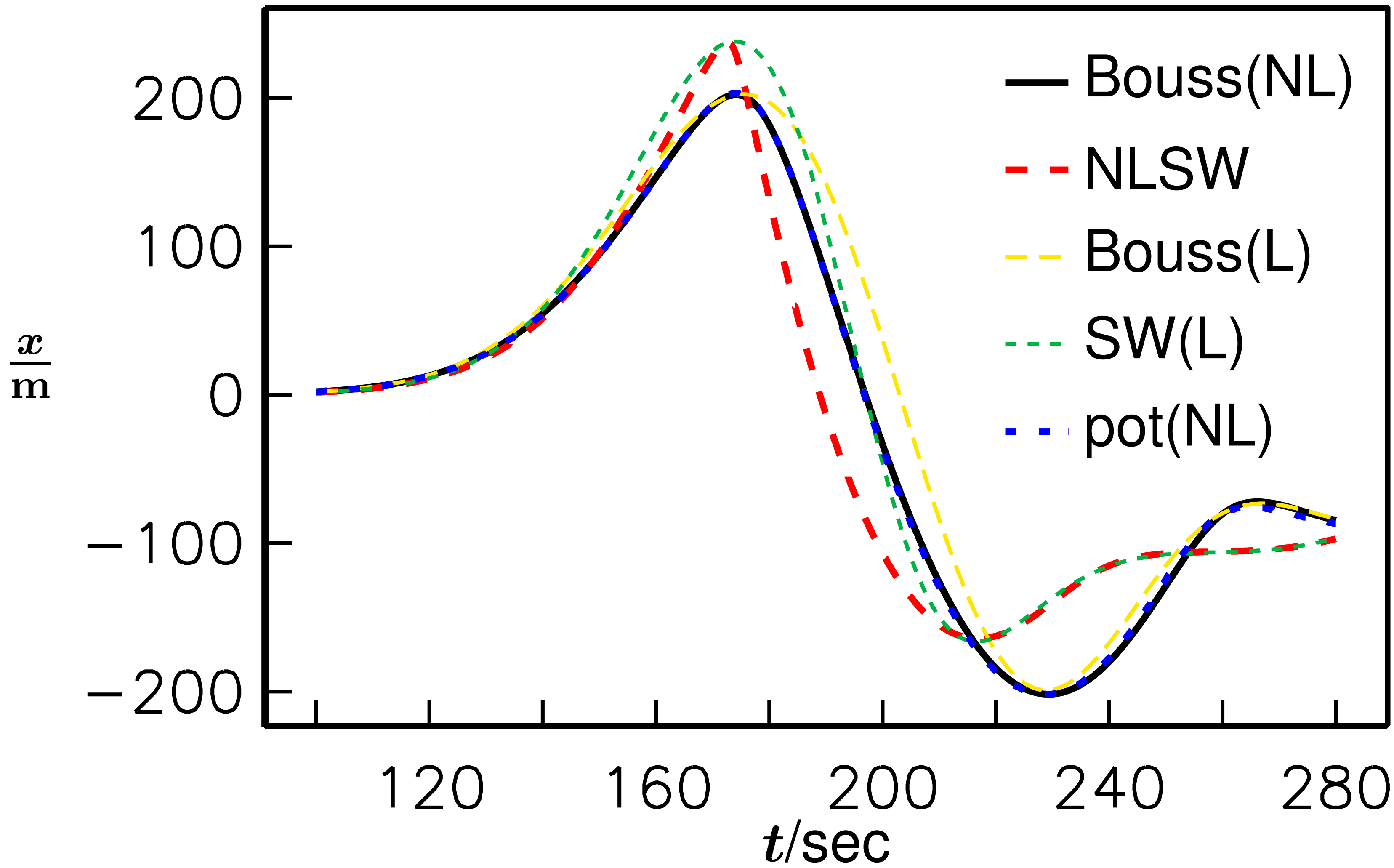
Surfaces at $t = 160.0$ sek. Comparison of models.

Local; $t = 175.0$ sec



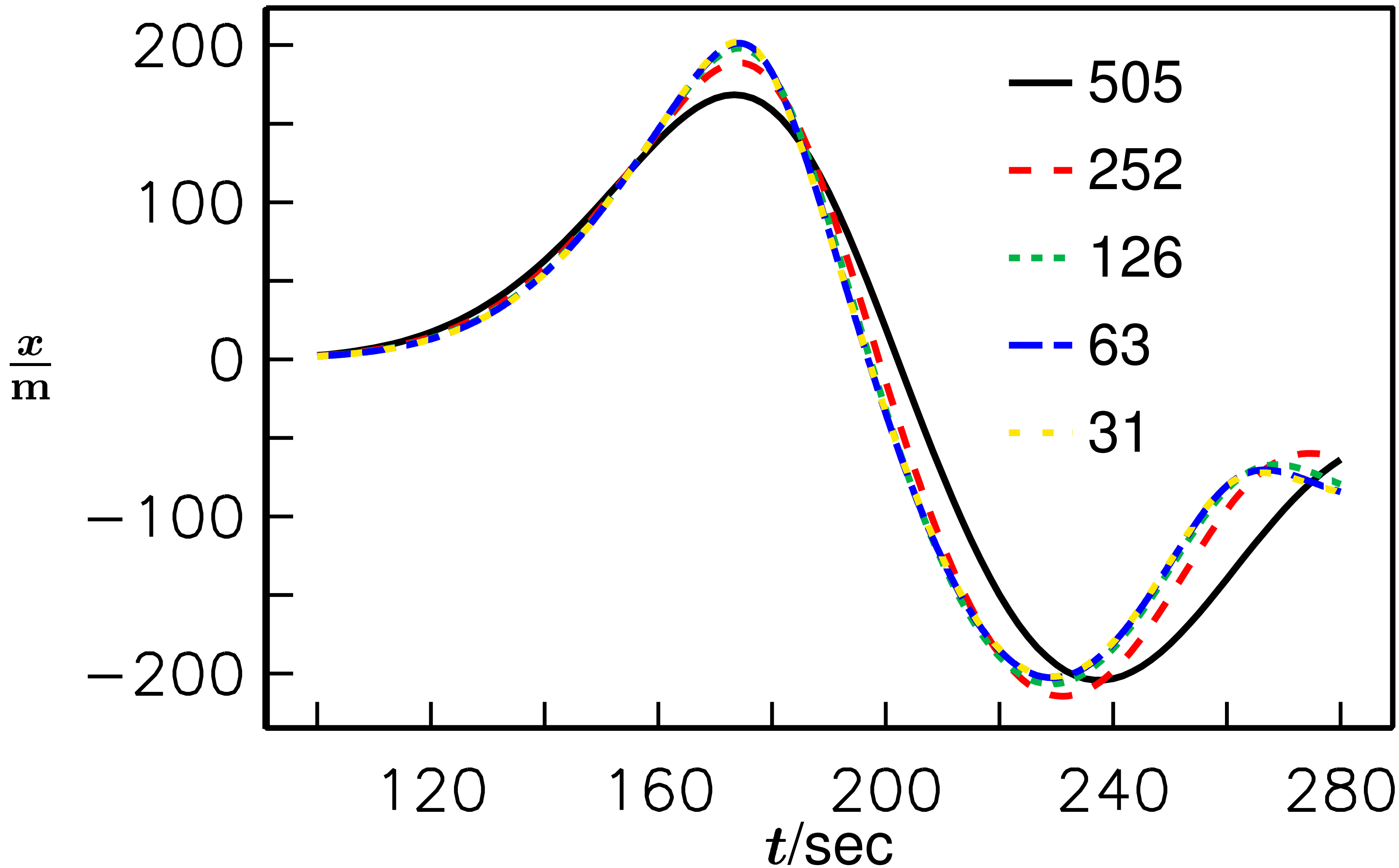
Surfaces at $t = 175.0$ sek. Comparison of models.

Shoreline motion



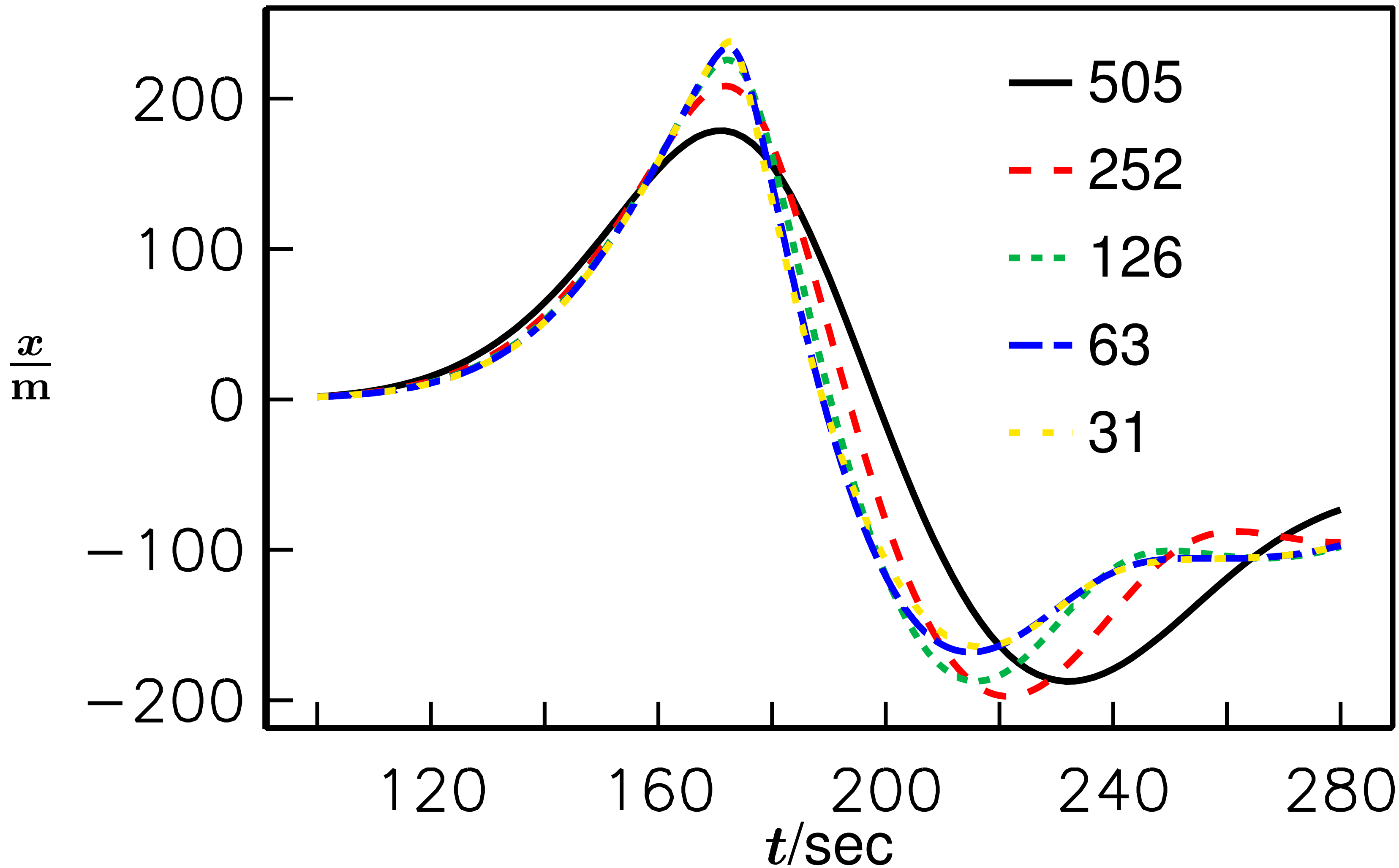
Inundation. Comparison of models.

Boussinesq; convergence of inundation



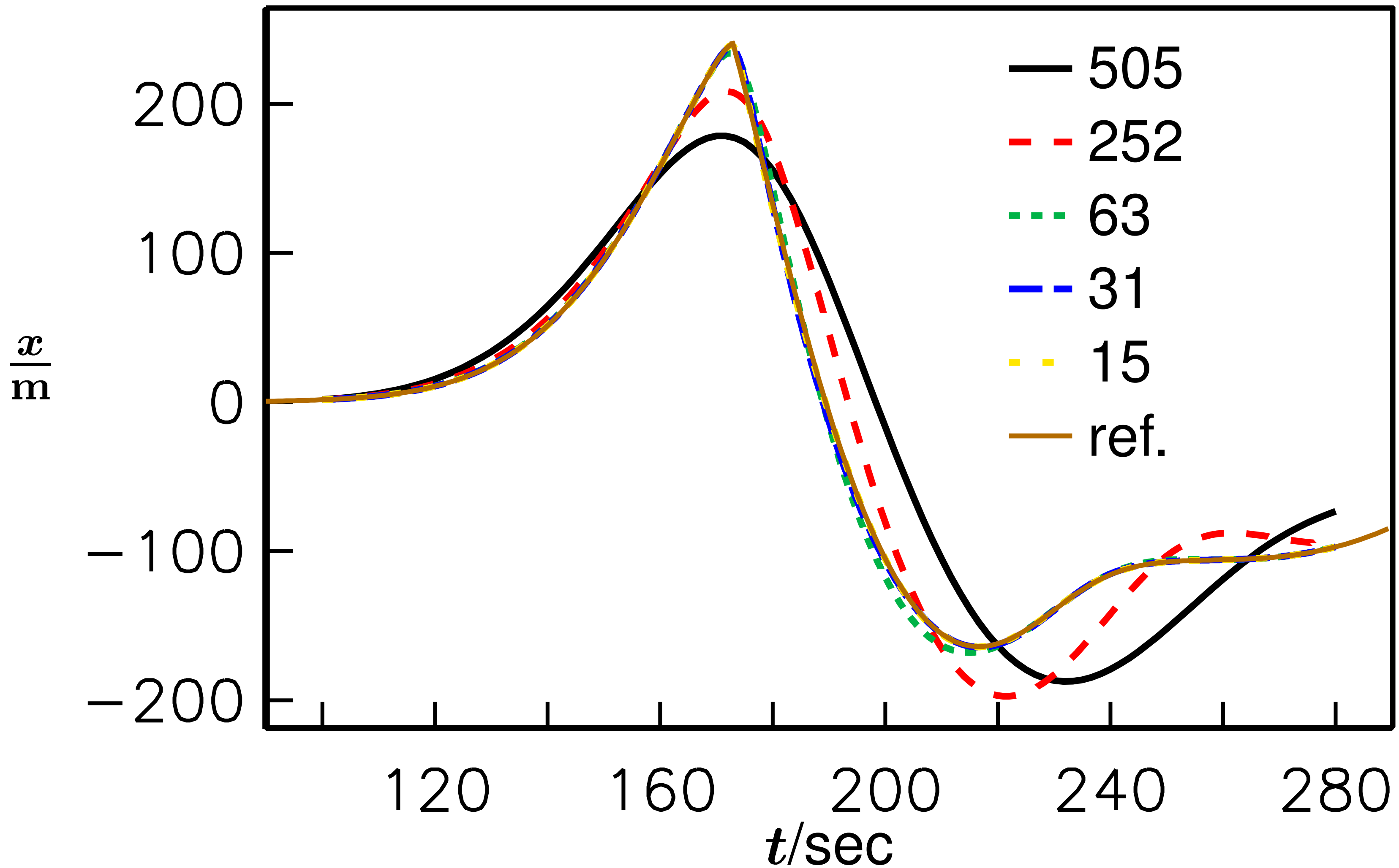
Bouss(NL). Curves marked by Δa (m)

NLSW; convergence of inundation



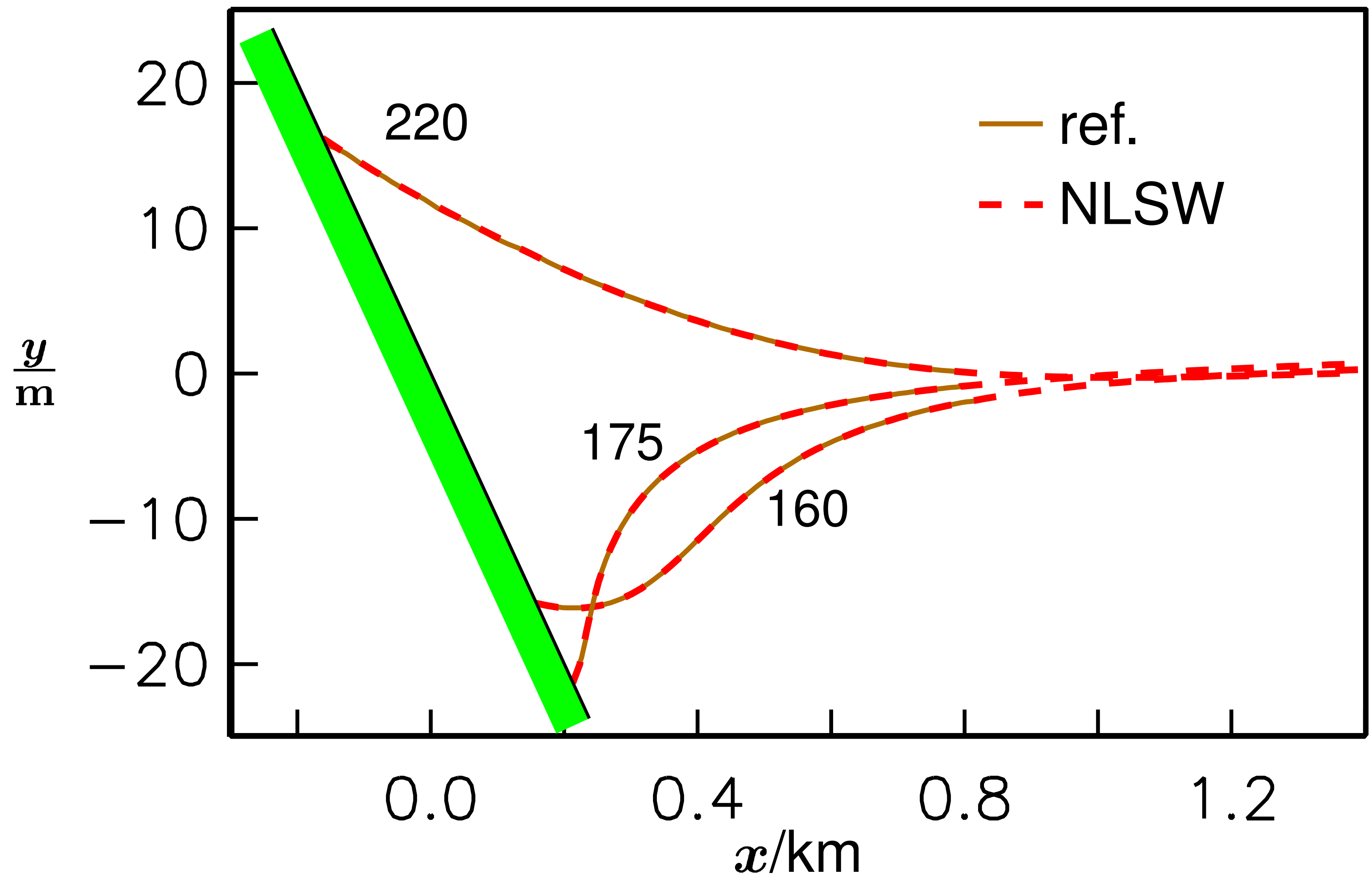
NLSW. Curves marked by Δa (m)

NLSW and reference solution



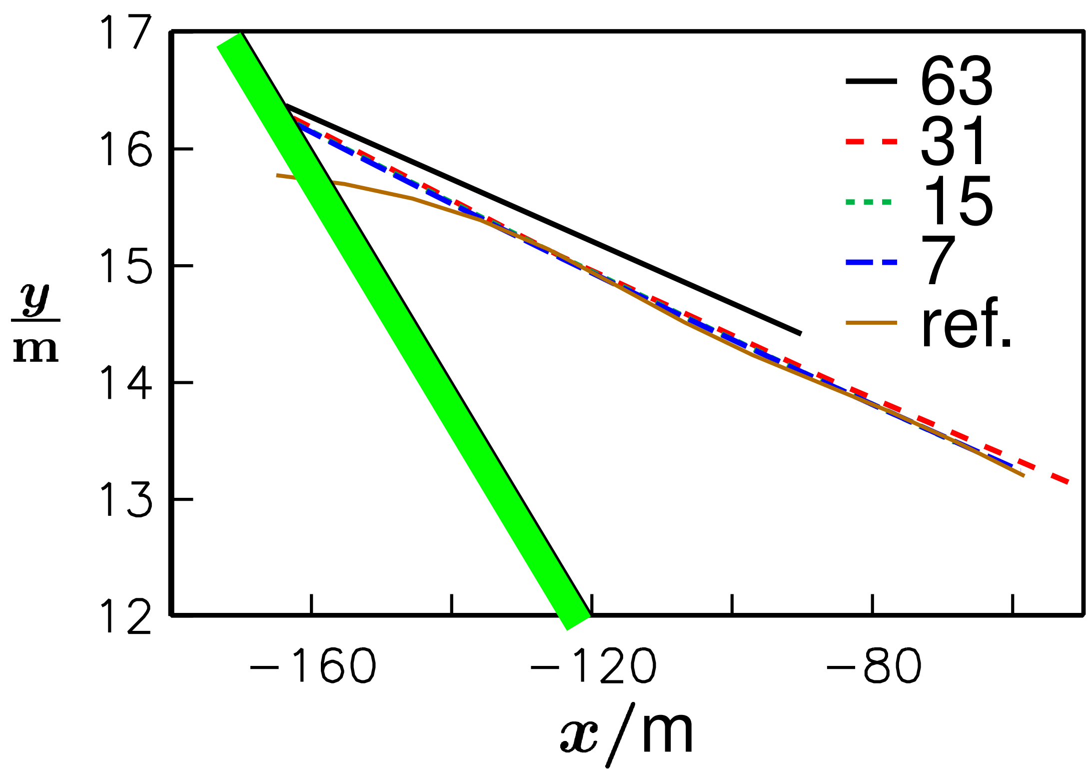
Analytic and NLSW. Latter marked by Δa (m)

Surfaces, NLSW and analytic



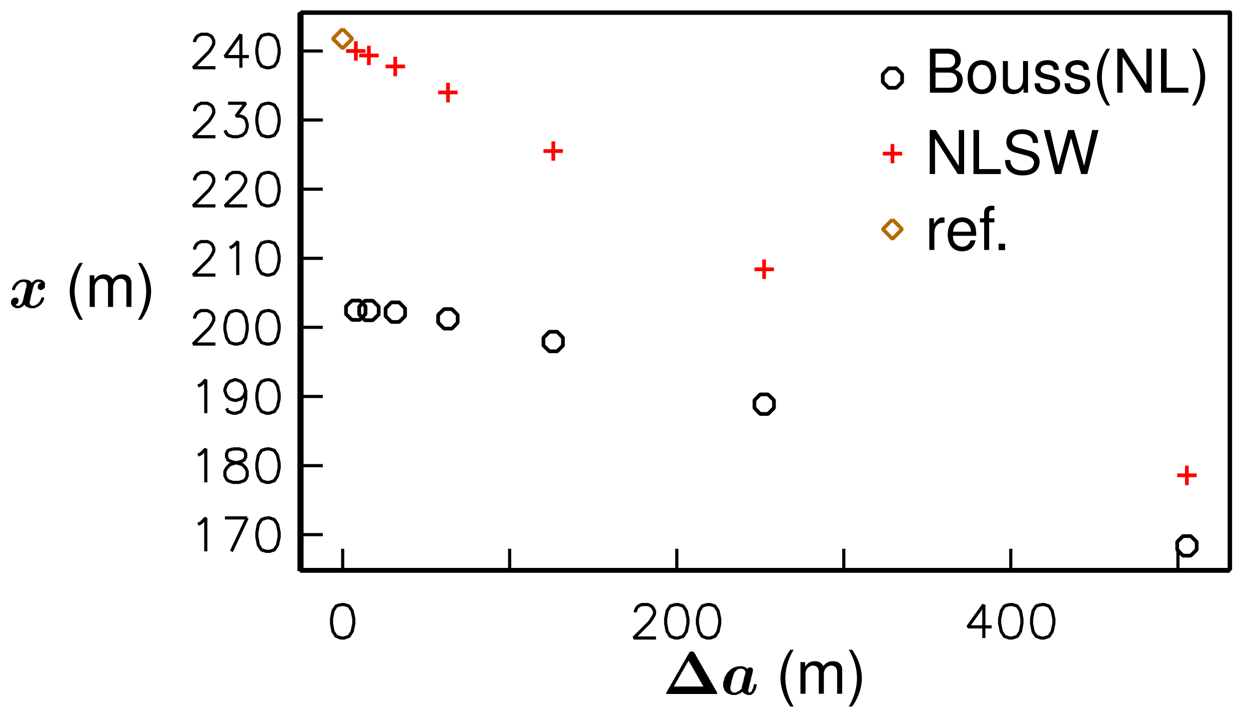
Comparison of NLSW ($\Delta a = 15$ m) and reference solution

Surfaces, closeup $t = 220$ sec



Analytic and NLSW. Latter marked by Δa (m).
Observe small scale features in reference solution

Maximum withdrawal; convergence



- NLSW $\Rightarrow$ virtually linear convergence in displayed range
- Bouss(NL) better (quadratic) convergence, fine resolution still needed

Concluding remarks 1

Wave models

1. Dispersion important for benchmark 1
2. Nonlinearity not important for shoreline extrema neither for shallow water models (as expected) nor dispersive models, but reduce duration of first withdrawal.
3. Boussinesq model in close agreement with full potential theory
4. Particular for NLSW: Large accelerations at max. draw-down. Reference solution becomes “weak” (non-differentiable) $\Rightarrow$ challenge for the numerics

Concluding remarks 2

Performance of numerics

1. Fine resolution required near shore
2. Close overall agreement between NLSW FDM and reference solution.
3. Discrete NLSW solution converges, but slowly, at max. draw-down
4. Minor unresolved features near shore, also in reference solution

Extras

The boundary integral model (BIM)

Related to high order technique of Dold (1992)

- Lagrangian particles traced at surface
- Cauchy's formula for complex velocity $(u - iv)$
- Cubic splines for field variables – solution is twice continuously differentiable
- "Moderately high order": less restricted at boundaries than Dold (1992)
  Special treatment of shoreline; invocation of analyticity

The Eulerian Boussinesq model

Standard Boussinesq equations expressed in terms of velocity and surface elevation. Nonlinear model, but used only in linear mode for benchmark problems. FDM, staggered in space and time
Variable grid: $\Delta x \sim \sqrt{gh}\Delta t$ for $h > h_m$

Inundation computed from η and extrapolated to shore ($x = 0$)
Consistent with

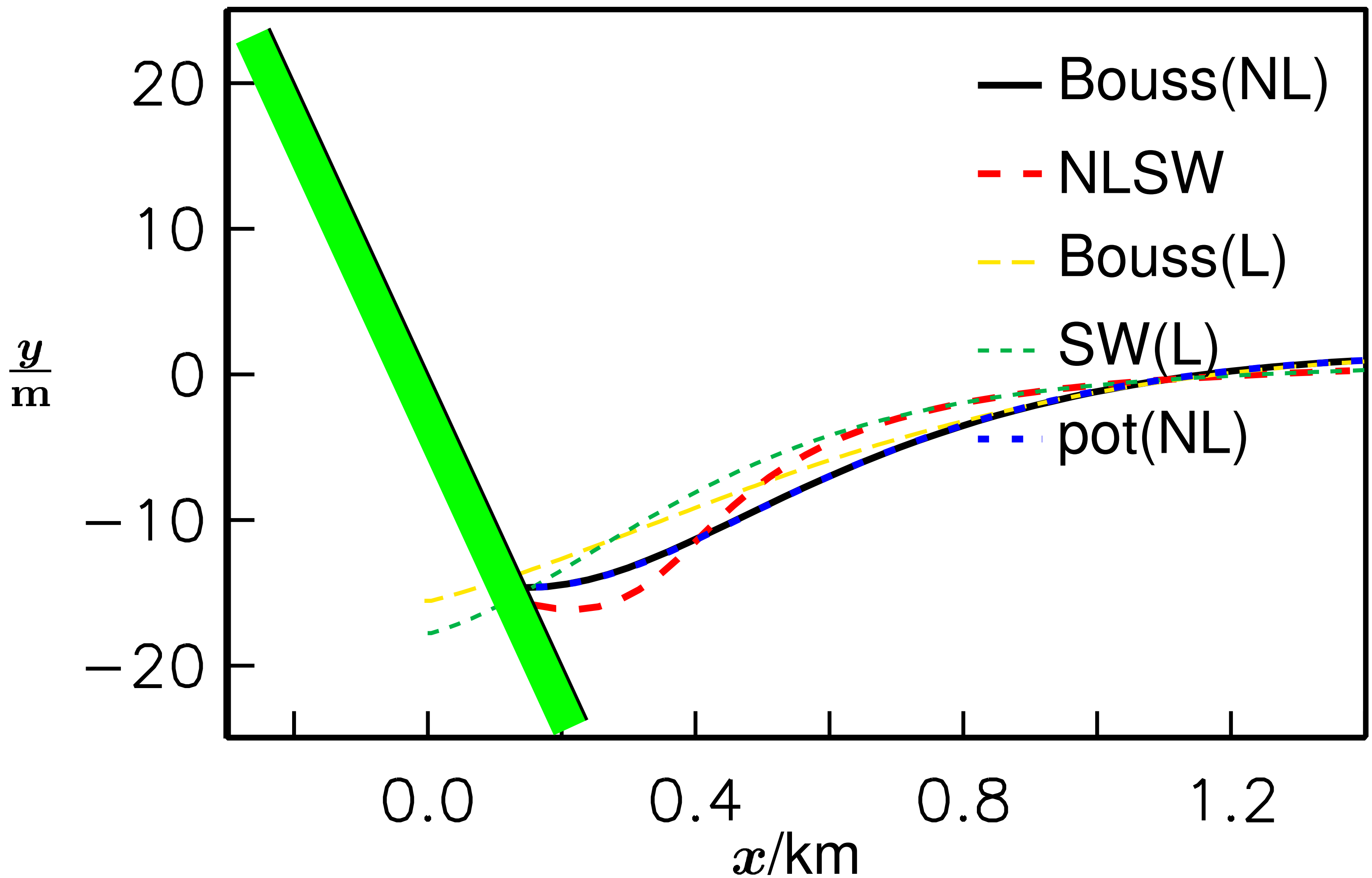
$$x_{\text{beach}} = \int_0^t u(0, \hat{t})\mathrm{d}\hat{t},$$

as inferred from the continuity equation.

The simulations

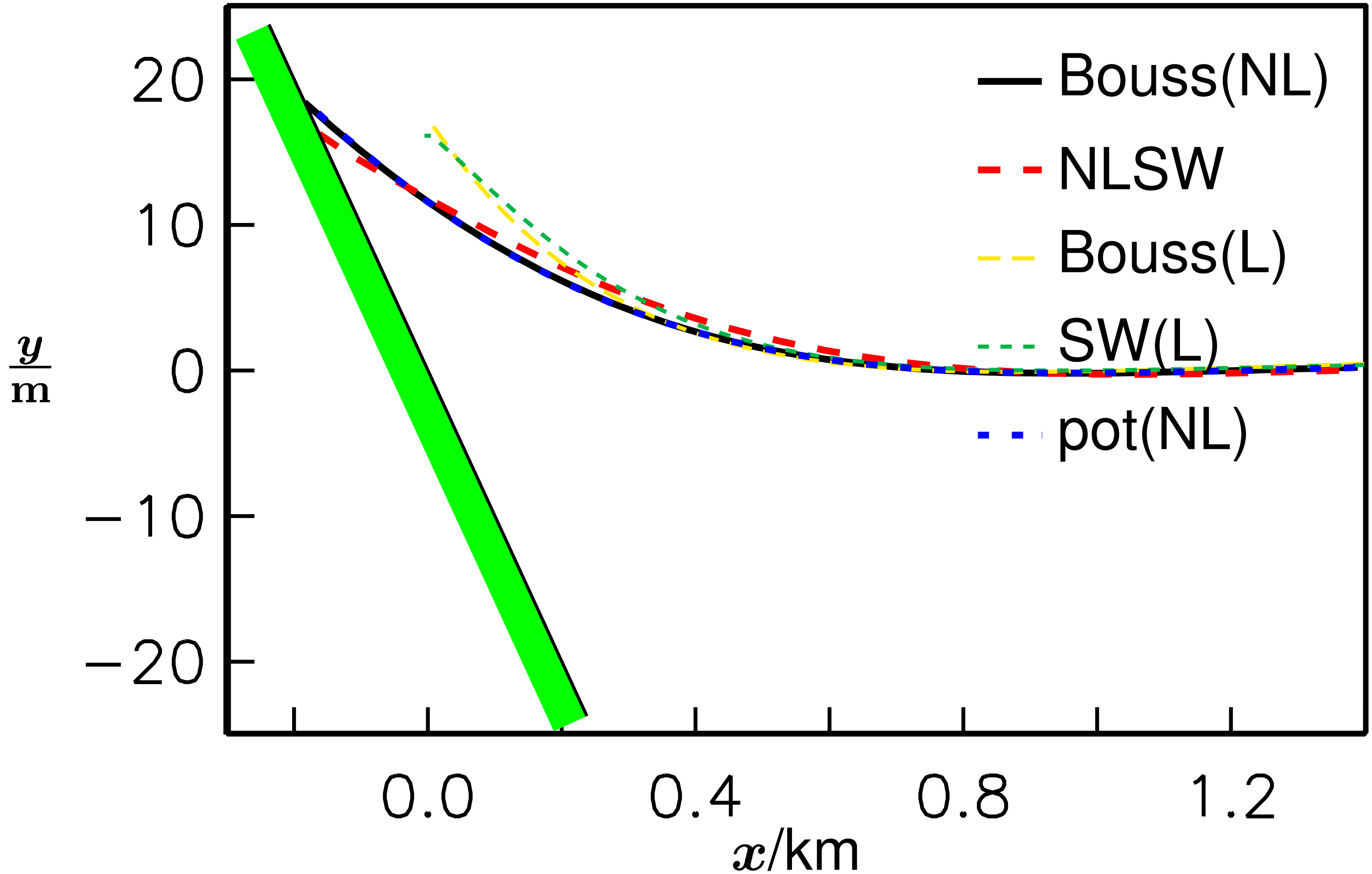
- Series of resolutions employed for each model
- BIM and Eulerian FDM: adaptive spatial refinement employed
- Lagrangian FDM: uniform resolution initially
- Semi-infinite domain used in reference paper Carrier, Wu and Yeh (2003)
  - Radiation conditions introduce additional errors and uncertainties – avoided.
  - Extra deep water region added. Domain $x < 50$ km not affected by farther boundary in 280 sec. Confirmed by tests.

Local; $t = 160.0$ sec



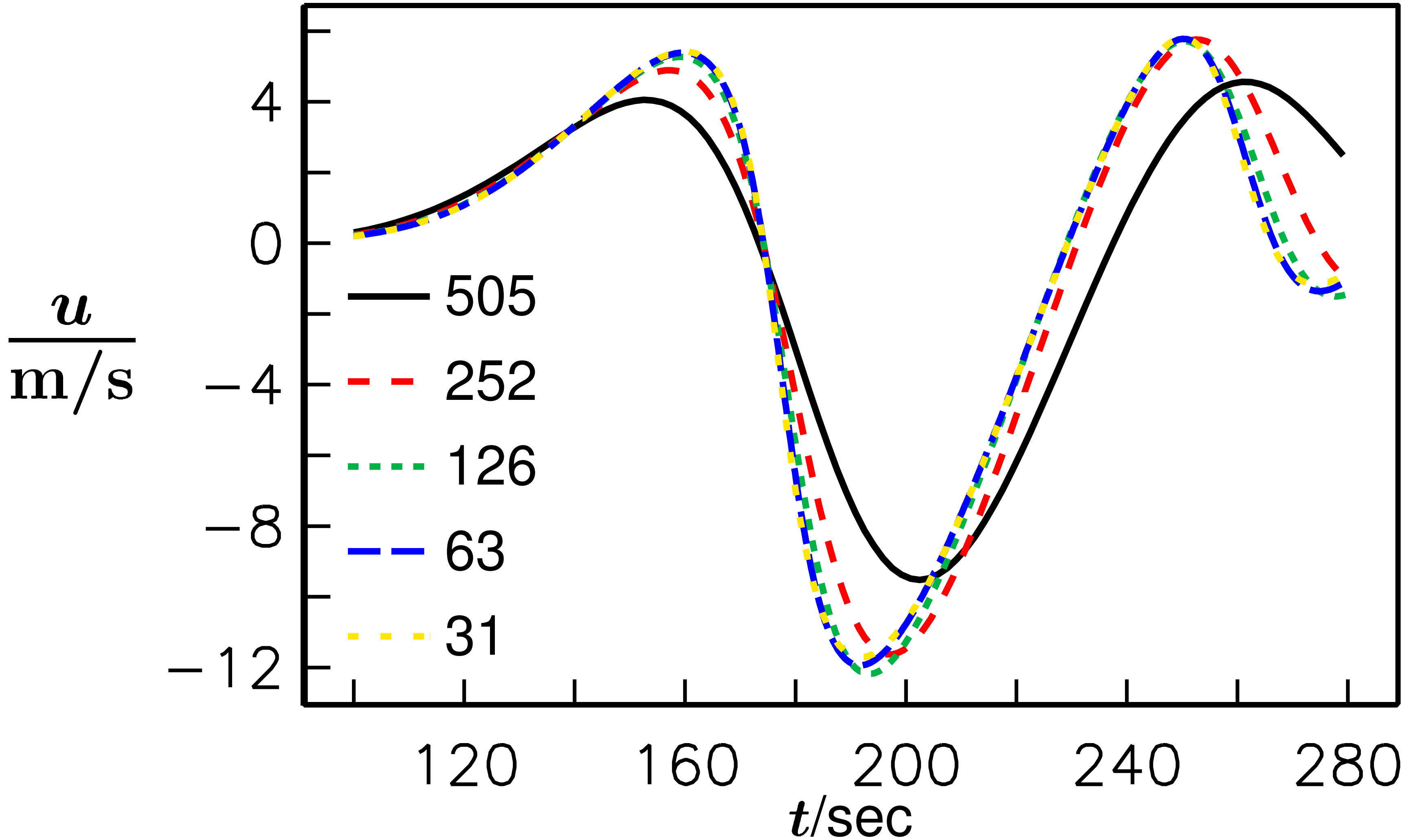
Surfaces at $t = 160.0$ sek. Comparison of models.

Local; $t = 220.0$ sec



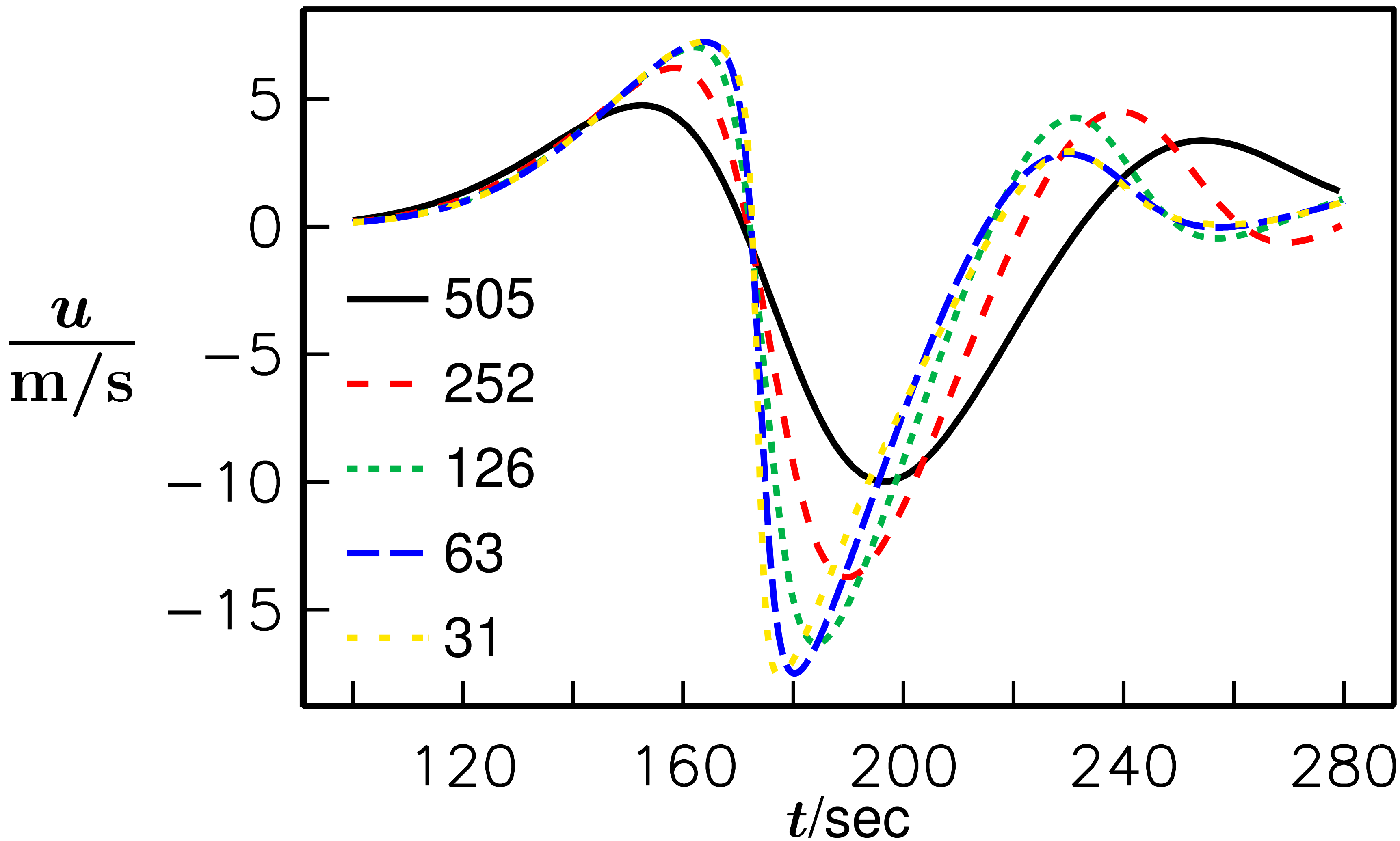
Surfaces at $t = 220.0$ sek. Comparison of models.

Boussinesq; shoreline speed



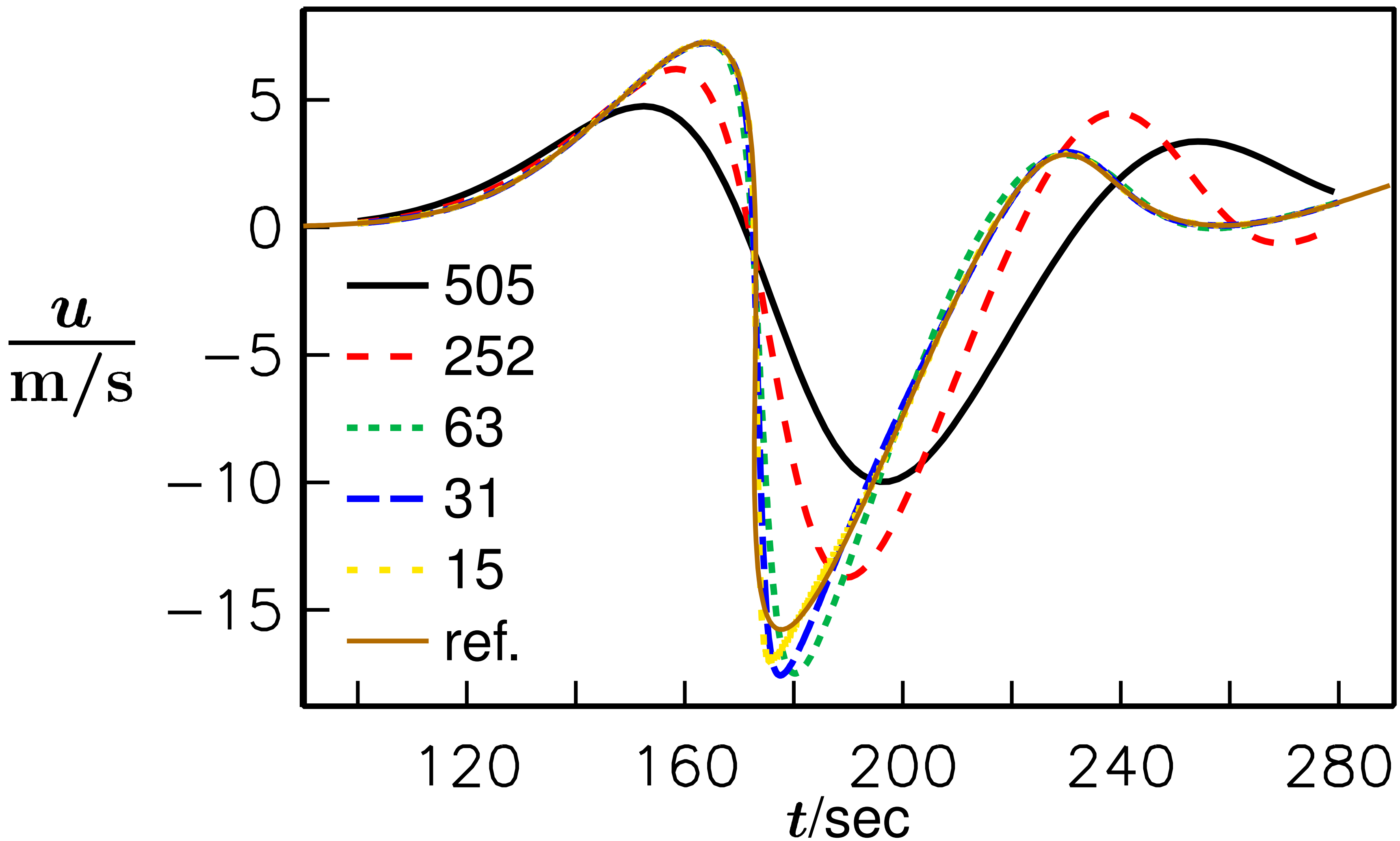
Bouss(NL). Curves marked by Δa (m)

NLSW; shoreline speed



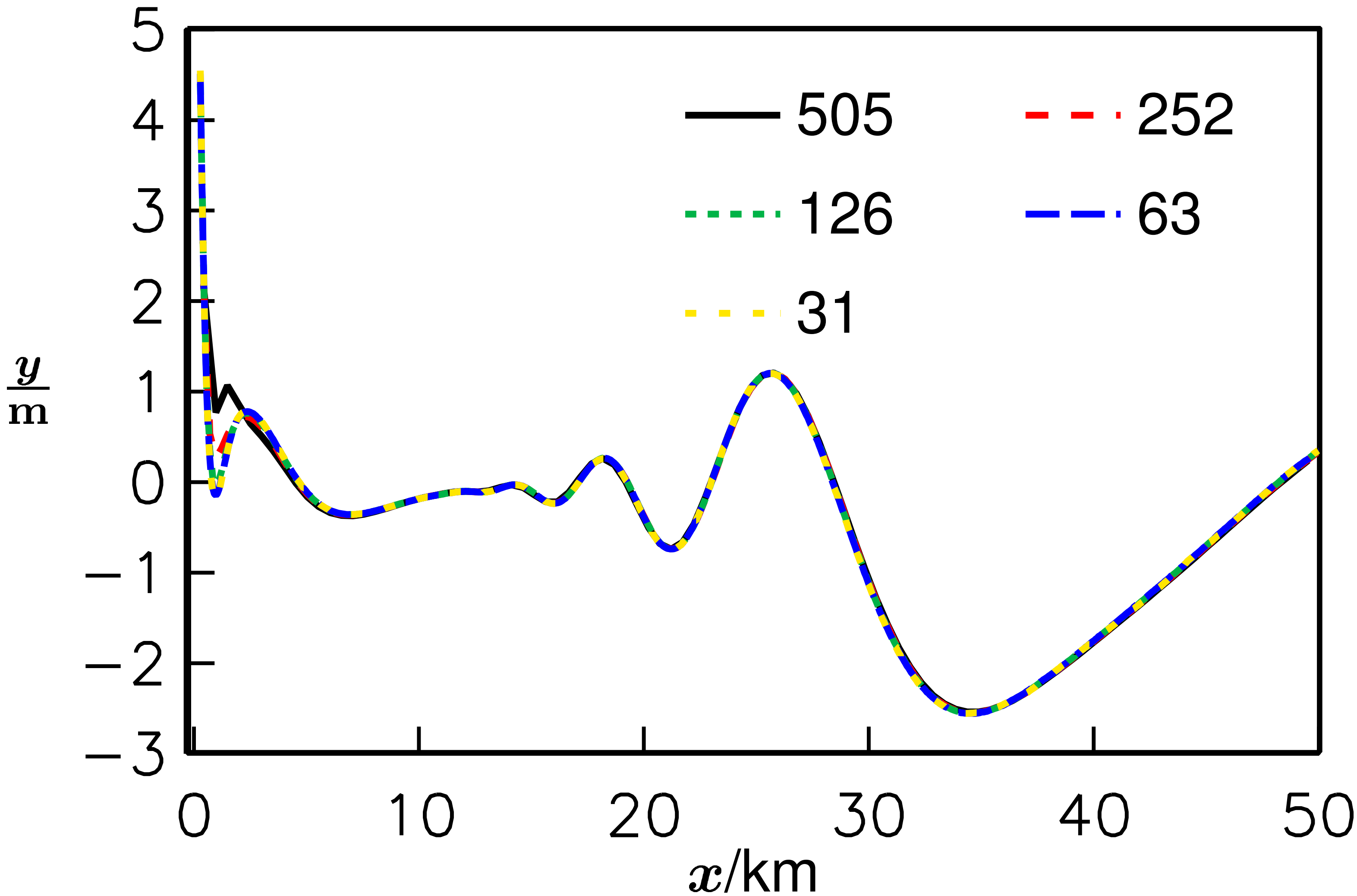
NLSW. Curves marked by Δa (m)

NLSW and reference solution



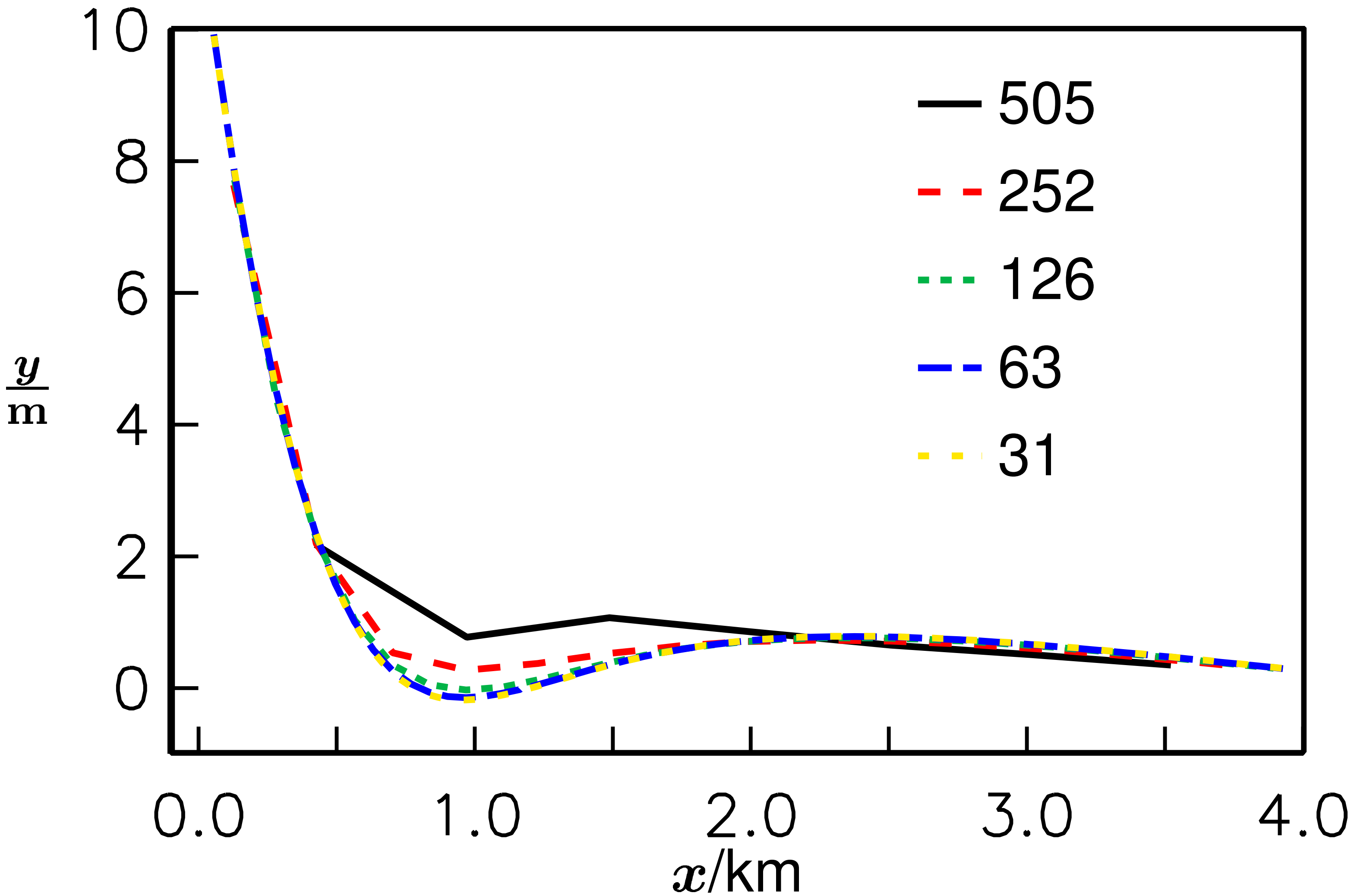
Analytic and NLSW. Latter marked by Δa (m)

Boussinesq; convergence $t = 220$ sec



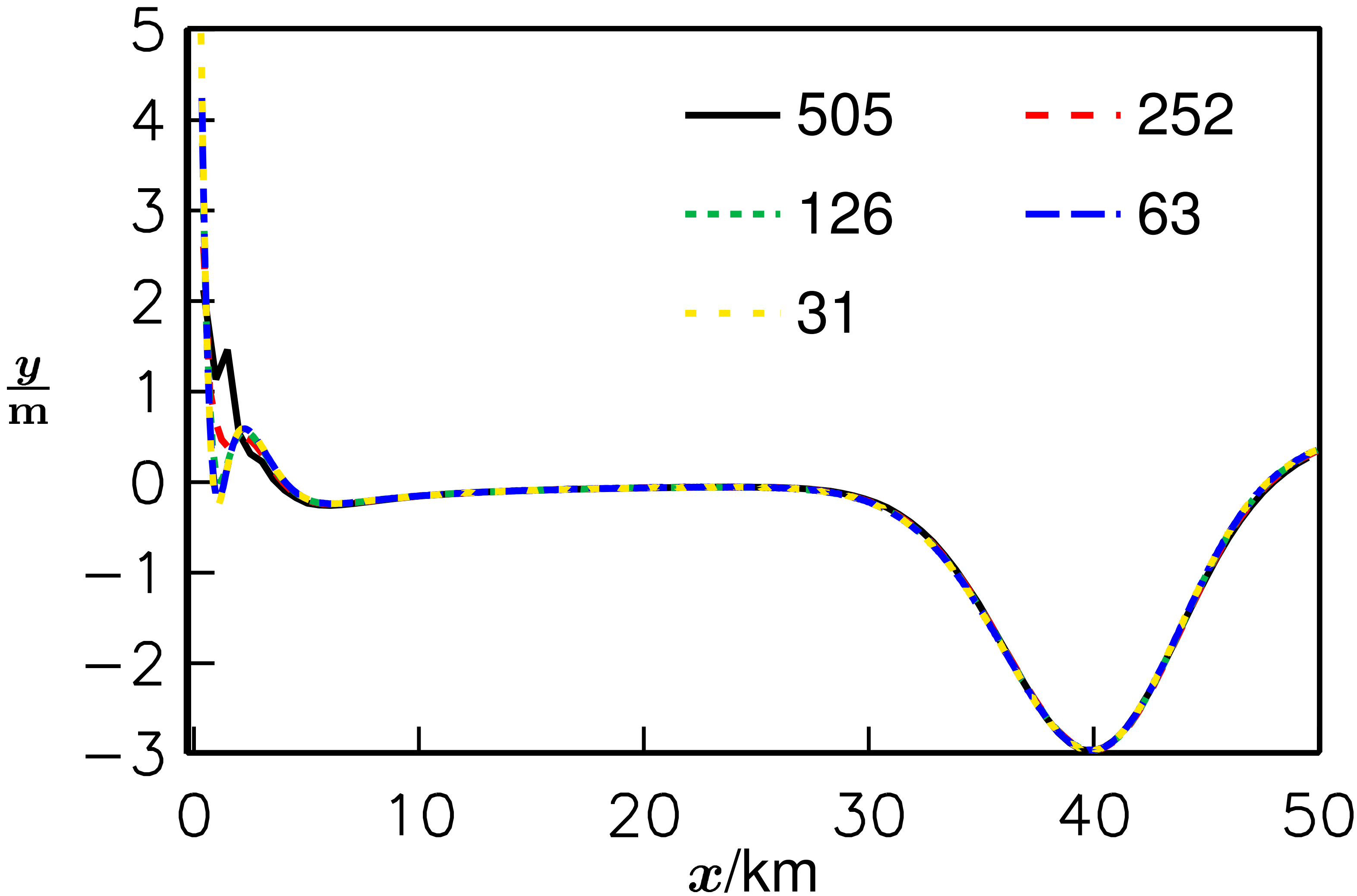
Bouss(NL). Curves marked by Δa (m)

Boussinesq; convergence $t = 220$ sec



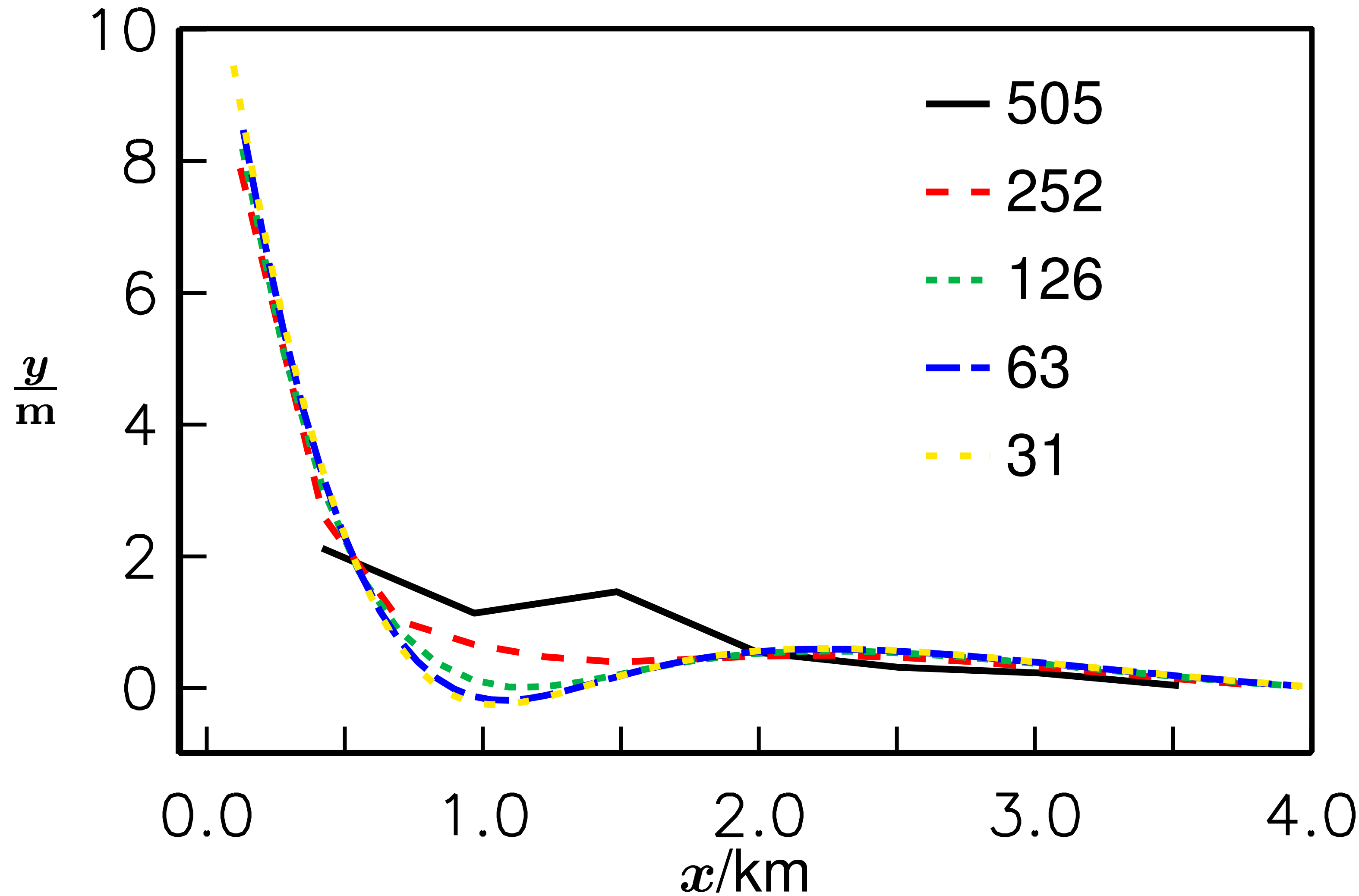
Bouss(NL). Curves marked by $\boldsymbol{\Delta a}$ (m)

NLSW; convergence $t = 220$ sec



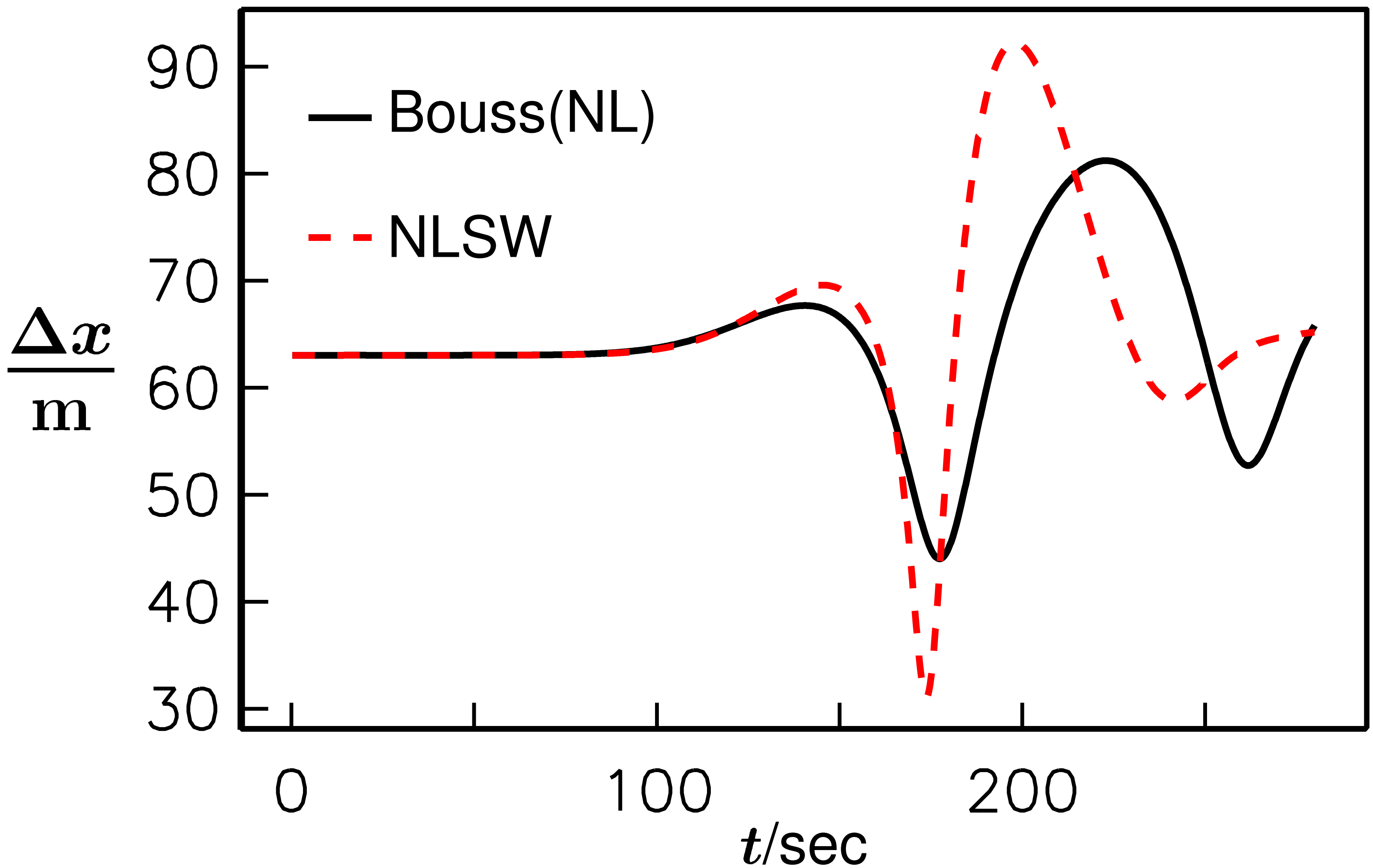
NLSW. Curves marked by Δa (m)

NLSW; convergence $t = 220$ sec



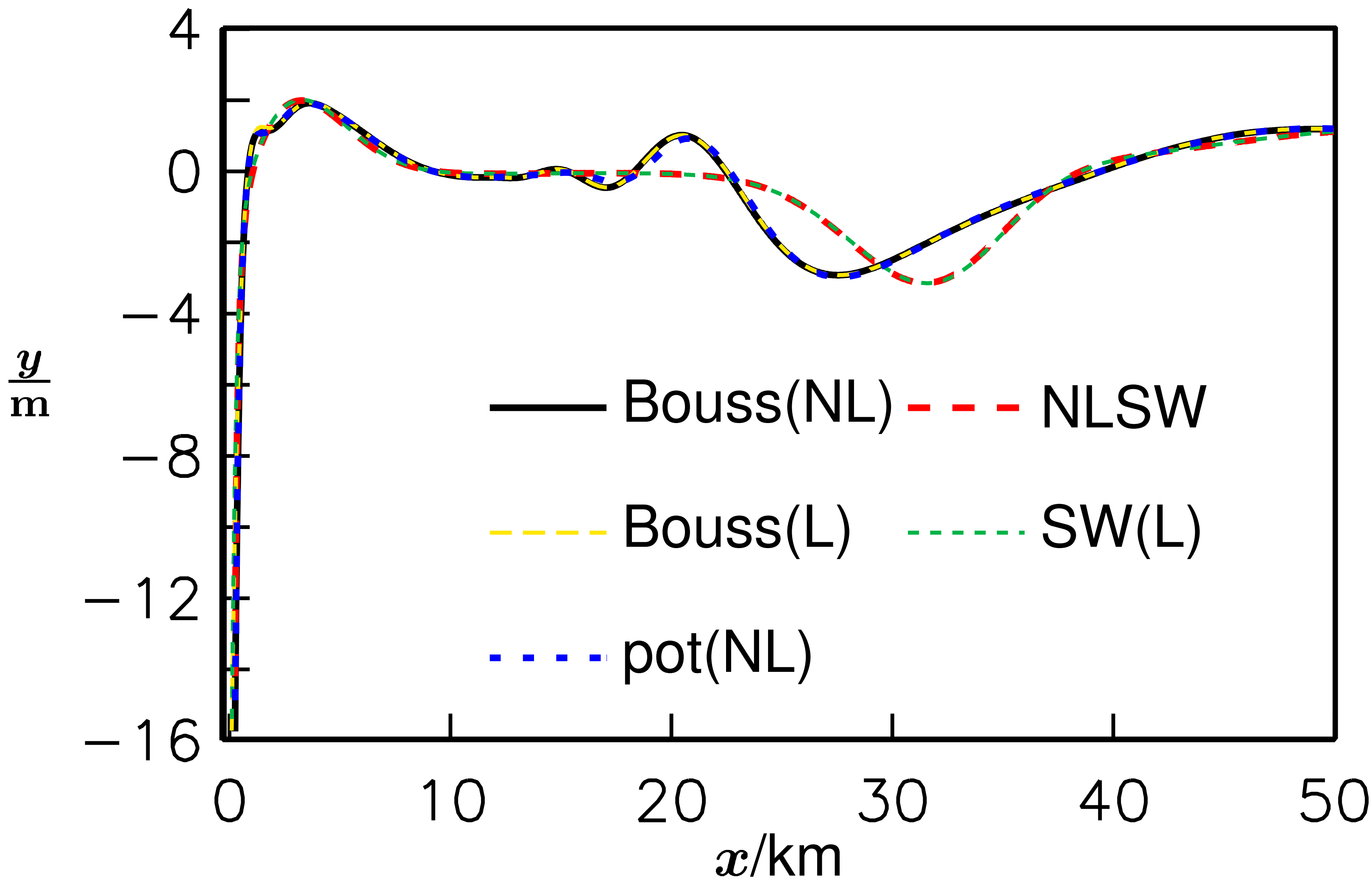
NLSW. Curves marked by Δa (m)

Grid stretching at beach



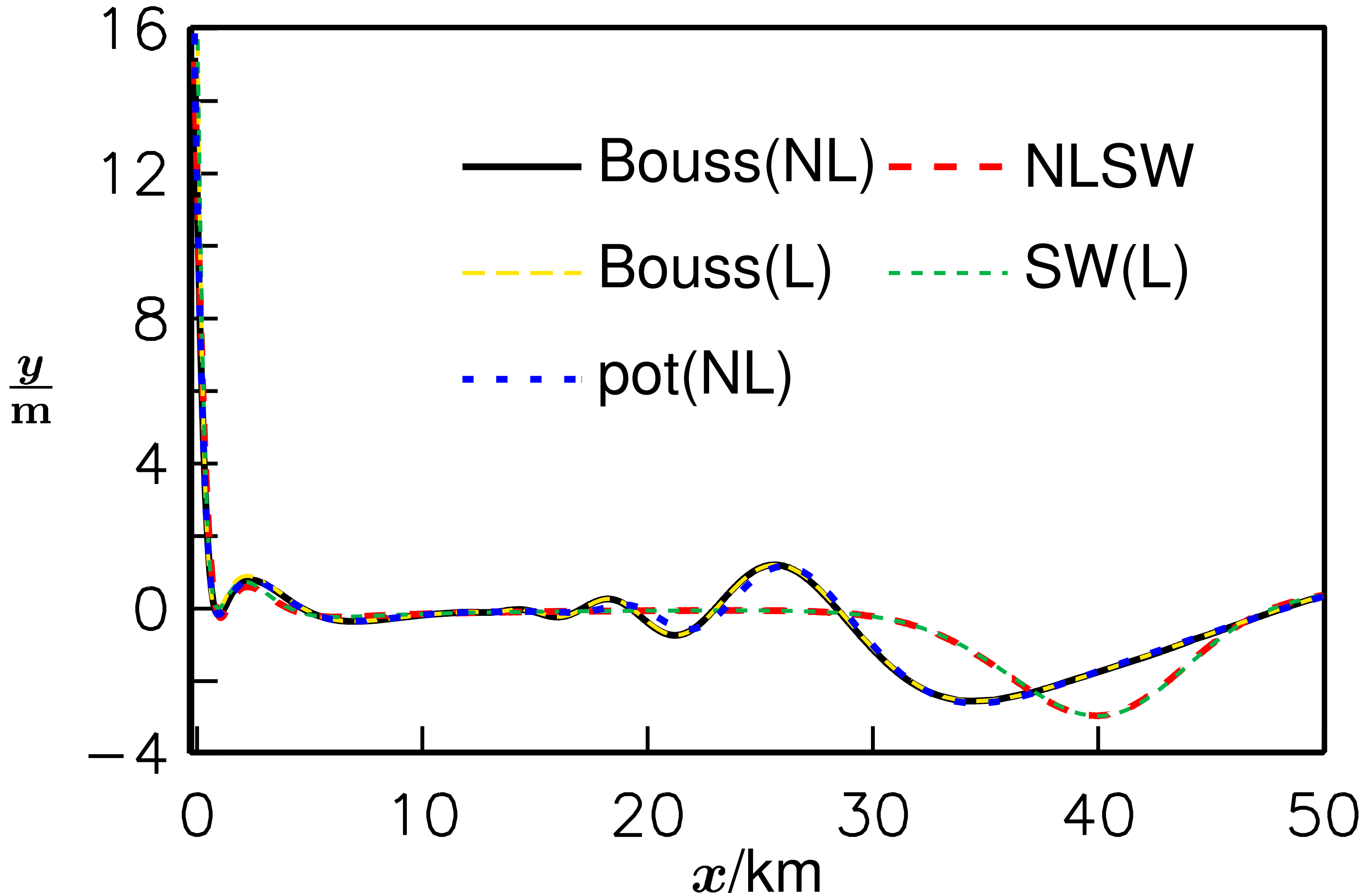
Grid spacing adjacent to beach.

Global; $t = 175.0$ sec



Surfaces at $t = \mathbf{175.0}$ sek. Comparison of models.

Global; $t = 220.0$ sec



Surfaces at $t = 220.0$ sek. Comparison of models.

References on models and related papers

- G. F. Carrier, T. T. Wu, and H. Yeh. Tsunami run-up and draw-down on a plane beach. *J. Fluid Mech.*, 475:79–99, 2003.
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