Modeling Wave Generation, Evolution, and Interaction with Depth-Integrated, Dispersive Wave Equations COULWAVE Code Manual Cornell University Long and Intermediate Wave Modeling Package v. 2.0

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1 Theoretical Background

The governing equations employed in this numerical model allow for the evolution of fully nonlinear (wave amplitude to water depth ratio = O(1)) and dispersive waves over variable bathymetry. Additionally, the generation of water waves by movement of the sea floor can be examined. The governing equations are given in the Appendix, which is a summation of the authors' analytical and numerical work concerning depth-integrated wave theory, submarine landslide modeling, and runup modeling. At this point in the development, modeling of submarine landslides cannot be performed by the user without some guidance by the author.

The user has a choice of using two different numerical schemes; a high-order finite difference (FD) method and a high-order finite volume (FV) method. The FD option is the "traditional" option, and is the scheme used by the original code. The FD option will provide very high accuracy and relatively quick computations. The downside of the FD option is that is it very sensitive to steep fronts and shocks, meaning that it is prone to crashing. The FV option uses a high-order, shock-capturing, approximate Reimann solver for the leadingorder flux terms, and is generally extremely stable and accurate (see Appendix for details).

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This scheme will add numerical dissipation for situations with poorly resolved shocks; where the wave is not resolved enough for the breaking model to turn on, the shock-capturing properties of the solver will maintain the steep front and dissipate energy numerically. The FV approach will also require 50 to 100 percent or more computational time, compared with the FD solution.

It is strongly recommended that the user choose the FV option. All examples provided with the source code use the FV option.

2 Numerical Background

The numerical model uses a fourth-order predictor-corrector scheme to march forward in time. Leading-order spatial derivatives are approximated to fourth order accuracy as well, while dispersive terms are second order accurate. The model is formally accurate to Δt^4 in time and Δx^4 , $\mu^2 \Delta x^2$ in space. The corrector segment of the procedure is implicit in time, and uses iteration to arrive at a solution. Details of the numerical model are given in the Appendix.

3 Files Included in Distribution

The following files are included in this package:

pCOULWAVE.exe : compiled Win-32 executable (from CVF, with MPICH libraries) pCOULWAVE.out : compiled Linux executable (from PGI, 64-bit with LAM MPI libraries) /source : COULWAVE source code; a series of Fortran 90 files /docs: COULWAVE documents_including:

/docs: COULWAVE documents, including:

/docs/USERS_MANUAL.pdf : Document including the theory, numerical approach, and description of how to use the model (this documents)

/docs/COMPILATION_INSTRUCTIONS.pdf : description of how to compile the code on a number of platforms

/examples : includes a number of directories corresponding to example setups. See EXAM-PLE_README.txt for a description of the included files in each directory. See the users manual for a description of each of the examples.

/examples/1D_solitary_wave_up_slope : recreation of Synolakis (1987) experiment

/examples/1D_regular_waves_up_slope : recreation of Svendson experiment

/examples/1D_irregular_waves_up_slope : TMA waves up a slope

/examples/1D_regular_waves_over_step : recreation of Dingman's step experiment

/examples/1D_irregular_waves_over topping : TMA waves over topping a levee with irregular foreshore

/examples/2D_solitary_wave_over_shoal : recreation of Conical Island solitary wave runup experiments

/examples/2D_irregular_waves_bar_break : 1D TMA waves over a nearshore bar with a break;

rip current generation /examples/2D_irregular_waves_real_beach : 2D TMA waves over real bathymetry

4 Compiling the Code

See the COMPILATION_INSTRUCTIONS.pdf file, included with this package distribution.

5 Using the Numerical Code

The code in written in Fortran 90, and has been compiled using Compaq Visual Fortran and Portland Group compilers. Other compilers, such as Intel, Lahey, and gfortran, have not been tested thoroughly.

This manual will not provide significant details on how to run MPI based code. To run as a serial simulation, one can run the executable as normal (e.g. double-click). To run in parallel, some type of MPI-based external command will need to be called, with command line options such as number of processors to use, nodes to use, etc. The most common format is:

mpirun -np 16 ./pCOULWAVE.out

which would run the executable "pCOULWAVE.out" on 16 processors.

A primitive user interface has been incorporated into the model, which allows the user to set the parameters of the numerical simulation without having to edit the source code and recompile. Therefore, any simulation can be run using the same compiled executable. The parameters/options governing the simulation are described here, in the order in which they appear when using the text interface. It is recommended that for the first few times running the program, the user follow along with this manual while running the executable. The user interface has been constructed such that all of the information contained in the user manual below is also given, in the form of long comments and instructions, in the user interface as well. This description will use the input files for the "2D_solitary_wave_over_shoal" example.

The first option menu is:

Loading simulation parameters from data file, sim_set you will be able to change the numerical parameters through menu choices.

****** Parallel/Domain Decomposition Setup ********
Number of Processors Selected: 1
ID 1: Automatic division of grids used
Number of divisions in x-dir: 1
Number of divisions in y-dir: 1
Are the above choices OK? - Enter ID to change or 0 for OK

By default, COULWAVE will attempt to divide up the total domain equally among the processors. For example, if the program is run on 16 processors, the automatic division would divide both the x and y range into 4 segments, providing 16 subdomains. If the total number of processors cannot be divided equally, the two closest factors are used, with the larger factor used to divide the x domain. For example, if 24 processors are used, the automatic division choice will use 6 segments in the x domain and 4 in the y domain.

This automatic division does not take into account the possibility that the length of the x and y domain may be unequal. Since the optimum subdomain shape is square, it is common that the user will want to specify the grid divisions manually. For example, if the total domain size was 800 points in the x direction and 200 points in the y, the user should choose ID 1 (enter 1 and hit return), choose the "Input domain divisions manually" option, and input 8 divisions in the x-direction. This will force the model to divide the x domain into 8 segments and the y into 2 segments, which gives the optimum 100 by 100 point subdomain.

When the user is satisfied with the domain decomposition, enter 0 and hit return to proceed to the next menu.

- ID 1: Surface wave evolution
- ID 2: 2D simulation
- ID 3: Fully Nonlinear Simulation
- ID 4: Arbitrary level approximation
- ID 40: One-layer model
- ID 41: Weakly Rotational Model
- ID 42: Finite Volume Solver
- ID 5: Solitary Wave Evolution

Are the above choices OK? - Enter ID# to change or 0 for OK

Through ID 1, the user can choose between "Surface wave evolution" and "Wave generation by submarine landslide." The landslide choice will create a simulation examining surface waves created by movement of the sea floor bottom. This option is not ready for general use.

Through ID 2, the user run either a 1D (transect or profile) simulation or a 2D. Note that the dimensions refer to horizontal coordinates; there is no vertical coordinate in this model.

Through ID 3, the user can choose either a Linear, Weakly Nonlinear (no nonlinear dispersive terms), or a Fully Nonlinear (with nonlinear dispersive terms) simulation. Compared to Fully Nonlinear choice, the Weakly Nonlinear approximation will significantly reduce CPU time (typically 15-25 % less for 2D), but may lead to large errors in the prediction of large amplitude waves. For example, if prediction of wave shoaling - wave height grows as water depth decreases - is desired, especially over mild slopes (< 1/20) the weakly nonlinear assumption should not be used as it tends to overestimate the wave height. Note that the Finite Volume Solver can only use the Fully Nonlinear equations.

Through ID 4, the user can choose the type of frequency dispersion model used. The choices are Arbitrary level (e.g. Nwogu approach), depth-averaged, and shallow water (non-dispersive). The Arbitrary level choice will make use of a set of governing equations based on evaluation of horizontal velocity at an arbitrary depth, given as $z = -\beta h$, where the optimum $\beta = 0.531$ for the one-layer (Boussinesq) model. The advantage of using the z_{β} method is that the wave and group velocities of higher wave numbers $(h/\lambda > .25)$ are more accurately described. The disadvantage is increased computational cost. When using more than one-layer, this option must be chosen. The evaluation levels for the two and more layer models are given in the Appendix.

The depth-averaged choice will have decreased CPU time, as compared to using Arbitrary level, but intermediate-depth waves may have significant phase and group speed errors. Therefore, a depth-averaged simulation should be utilized if the user is fairly certain that all wave numbers will be small, i.e $h/\lambda < .2$, or if computational speed is important (typically 5-15 % less for 2D). Note that the Finite Volume Solver can not use the depth-averaged model.

The shallow water wave equations are non-dispersive, and are only accurate for very long waves. Note that there are numerical packages in existence that will solve the shallow water equations many times faster than this program.

Through ID 40, the user can choose to use the one-layer (conventional) Boussinesq-type model, or the two-layer model of Lynett and Liu (2004). Note that the Finite Volume Solver can not use the two-layer model.

Through ID 41, the user can choose whether to run an irrotational or weakly rotational simulation. The weakly rotational model includes both weak horizontal and vertical vorticity (See Kim et al paper in Appendix). Note that the Finite Volume Solver can not use the irrotational model.

Through ID 42, the user can choose between the Finite Difference and Finite Volume solver. The FD option is the "traditional" option, and is the scheme used by the original code. The FD option will provide very high accuracy and relatively quick computations. The downside of the FD option is that is it very sensitive to steep fronts and shocks, meaning that it is prone to crashing. The FV option uses a high-order, shock-capturing, approximate Reimann solver for the leading-order flux terms, and is generally extremely stable and accurate (see Appendix for details). This scheme will add numerical dissipation for situations with poorly resolved shocks; where the wave is not resolved enough for the breaking model to turn on, the shock-capturing properties of the solver will maintain the steep front and dissipate energy numerically. The FV approach will also require 50 to 100 percent or more computational time, compared with the FD solution. It is strongly recommended that the user choose the FV option.

Through ID 5, the user specifies the wave condition. The choices are solitary wave, sine (regular) waves, or a wave spectrum. The solitary wave profile is the analytic solution to the weakly nonlinear equations. Therefore, larger amplitude waves will not initially be of permanent form when using this option in the Fully Nonlinear model. If the wave spectrum choice is made, the user must create the input spectrum with the provided spectrum_1D.m (for 1D spectra) or spectrum_2D.m (for 2D spectra) in Matlab as a pre-processing step. These scripts will create a spectrum.dat file that will be read by COULWAVE. Using this script file is discussed in the later section "Creating a Wave Spectrum in Matlab.".

Solitary Wave Evolution

ID 1. Wave height:(m) 5.7999998E-02

ID 2. Initial/characteristic depth:(m) 0.3200000

ID 3. Initial location of crest of soliton (m) 10.00000

ID 10. Incident angle of waves 0.000000E+00

Are the above choices OK? - Enter ID# to change or 0 for OK

The above is the menu for the solitary wave simulation, which should be fairly straightforward to modify.

If one was running a sine wave simulation, the menu would appear as:

If one was running a spectral wave simulation, the menu would appear as:

And finally, if one was running a simulation driven by an input time series, the menu would appear as:

The next option menu is:

ID 99. Load Topographical Data from Files

Are the above choices OK? - Enter ID# to change or 0 for OK

In this tutorial, the bathymetry / topography surface has been generated through a pre-processing step, using the included bath_loc.m Matlab script file. The Matlab script

"bath_loc" will create topography data from local files (again, see the file "bath_loc.m" for details). The files created by these scripts are: "x_topo.dat, y_topo.dat, f_topo.dat, size_topo.dat", and must be located in the same directory as the executable when running the program. For more information on how to create a topography using the Matlab scripts, see the later section "Creating a Bathymetry in Matlab." Note that the resolution of this externally generated surface is independent of the resolution used by the hydrodynamic model. COULWAVE will interpolate the depth grid onto the hydrodynamic grid using bi-linear interpolation.

The user can also create a depth profile by inputting a series of (x,depth) nodes (maximum of seven). Enter 99 and hit return, then choose "Specify Profile by Giving Location/Depth Nodes" and input the desired profile.

The next option menu is:

ID 1. Simulation time in seconds: 15.00000 ID 2. Time increment to write to file (s): 0.1000000ID 3. Number of grid points per wavelength: 50 ID 4. Courant number = $dx/dt/c_0$: 0.2500000 0 =Do Not Use Sponge Layer 1 =Use Sponge Layer Left Wall **Right Wall** Bottom Wall Top Wall ID 9 ID 10 ID 11 ID 12 0 0 0 0 ID 16. Display screen data at time step interval: 1

Are the above choices OK? - Enter ID# to change or 0 for OK

Here the user can change the physical simulation time, the time increment to write surface data to file, the number of grid points per wavelength, the time step through the Courant number, side wall boundary conditions, and the screen information display interval.

The surface data that are written to file include the free surface elevation, the horizontal velocity vectors near mid depth and at the free surface, the wet/dry cell assignment, the eddy viscosity, and the vertical vorticity evaluated at the free surface. These output files are written in binary, but even so, for large grid simulations files can become excessively large if a small time increment is chosen.

Points per wavelength is based on the 95% solitary wavelength, or the sine wavelength, or the peak period wavelength for spectrum simulations.

The Courant number will determine the time step used in the model, and value of 0.5 will typically yield stability and convergence, but for simulations with highly nonlinear waves, a value as low as 0.1 may be required for stability. It is recommended that a value of 0.25 is used by default.

For absorbing boundary conditions, the user should turn the sponge layers on by changing the appropriate ID to 1. If the ID is zero, that wall will act like a completely reflecting vertical wall. Note that the left wall corresponds to x=0, the right wall to $x=x_-$ end, the bottom wall to y=0, and the top wall to $y=y_-$ end. The sponge layer used here absorbs both mass and energy, and has shown to be an excellent absorber of waves of all types, with negligible reflection.

ID 1. Number of time series to write to file 0 ID 1000. Not performing spectral analysis on time series

Are the above choices OK? - Enter ID# to change or 0 for OK

Here, the user can specify locations, or stations, to write time series. Included in each time series is free surface elevation, x-component velocity at mid depth, and y-component velocity at mid depth. The user can also choose to have COULWAVE perform spectral analysis on each of the time series, which provides mean and significant properties for the written variables. Note that the time series locations are stored in the ASCII file ts_locations.dat. For simulations where many time series are written, it is often easier to edit this file outside of the user interface, in, for example, Excel or Matlab.

The next menu is:

- ID 1. Wave Breaking Model implemented.
- ID 3. Bottom friction included.
- ID 4. Roughness height, ks (m): 9.9999997E-05
- ID 5. Coefficient for subgrid eddy viscosity: 0.2000000

Are the above choices OK? - Enter ID# to change or 0 for OK

This menu covers the dissipation parameterizations. The first choice allows the user to turn the breaking model off or on. Description of the breaking model is included in the Appendix. The user can also turn bottom friction off or on. In the FD model, the bottom friction is characterized through a Mannings roughness factor. In the FV model, the user should input a roughness height, similar to that needed to use the Moody diagram. Lastly, the subgrid eddy viscosity for the horizontal Smagorinsky eddy viscosity is needed. The default for this parameter is 0.2, although it is expected to lie between 0.05 and 0.2. The higher the coefficient, the larger the dissipation. The last menu is:

- ID 2. Width of sponge layer, in wavelengths 1.0000000
- ID 3. Corrector stage convergence error 4.9999999E-06
- ID 5. Max # of allowable iterations in corrector loop 15.00000
- ID 6. Min # of iterations in corrector loop 2
- ID 9. Smoothing depth profile using 4-point filter.
- ID 11. First sponge layer coef 10.00000
- ID 12. Second sponge layer coef 9.9999998E-03
- ID 13. Shoreline can move? 0=Yes, 1=No 0

Are the above choices OK?-Enter ID# to change or 0 for OK

The default parameters are not recommended to be changed.

6 Error Messages and Descriptions

Error: Maximum number of iterations reached - corrector did not converge

Reason: The iterative corrector loop did not converge to the corrector stage convergence error within the specified maximum number of iterations allowed.

Solution: This error is either telling the user that there is some instability in the solution or that the values of the variables is so small that round-off errors and word length are playing a role. For the later to be the culprit, the error would appear during the early time steps of a sine or spectrum wave simulation. If this is not the case, plot the output in Matlab, and examine the free surface for short wave instabilities, and see if they are linked with rapidly changing bathymetry. If this appears to be the case, then smooth the bathymetry. If there is apparently no cause for either the instabilities or the high number of iterations, contact the author my email for assistance at plynett@tamu.edu.

SIMULATION ERROR - OVERFLOW

Reason: There is nothing occurring in the simulation. The likely cause is that the simulation overflowed (a physical value was assigned a value of "infinity"), whereby everything in the simulation is forced equal to zero. This error is compiler specific - some compilers will force everything equal to zero, some will display an error "overflow" and stop the simulation. An overflow is usually caused by an instability in the somewhere in the domain.

Solution: Plot the output that you have in Matlab, and look for any locations that show instabilities. Look for locations with very steep slopes. Steep slopes or corners should be slightly smoothed to eliminate the large first and second derivatives of water depth. If the simulation contains extremely high velocities (Froude numbers greater than 1), then lower the time step and rerun the simulation. If there is apparently no cause for the overflow contact the author my email for assistance at plynett@tamu.edu.

7 Known Problems and Deficiencies

Most simulation crashes occur due to very high Froude number flows, whether that is due to very high velocities or very thin flows. The dissipation models included sometimes do properly model these flows, causing instabilities. The best current work arounds are to increase the bottom friction, increase the eddy viscosity coefficient, decrease the grid resolution, and to decrease the Courant number.

Also, with bottom slopes and curvatures that are extremely large, instabilities can result. This is particularly the case with the FD model, where the maximum slope that one can expect to get a stable result is approximately 0.2, and the maximum curvature is 1.0. For the FV model, one can generally use values one order of magnitude larger.

With the FV model, the user should be careful when choosing to use the flux limiter option. While turning this option on will make for an extremely robust model, it can also add significant numerical dissipation, particularly for steep shoaling waves. To see the impact of the limiter, the example "1HD: Breaking regular wave runup and rundown" is a good testing simulation.

8 Creating a Spectrum in Matlab

The included Matlab scripts "spectrum_1D.m" and "spectrum_2D.m" can be used to create an input spectrum to be used in the numerical simulation. The "spectrum_1D.m" file will create a spectrum in one-horizontal dimension. This file creates a shallow-water based TMA spectrum. The user must edit the first few lines of "spectrum_1D.m" to input the peak frequency, significant wave height, and the depth of these waves. The output from this script file will be a data file called "spectrum.dat", which must be located in the same direction as the executable in order to be loaded by the simulation. The file "spectrum_2D.m" creates a two-horizontal dimension spectrum. The two-dimensionality is setup as:

$$S(f,\theta) = S(f)h(\theta) \tag{1}$$

where the $h(\theta)$ directional function used in the Matlab file is a simple \cos^2 spreading function. Additionally, the user can load a local energy density spectrum data file using the given Matlab files.

9 Creating a Bathymetry in Matlab

The included Matlab script "bath_loc.m" can be used to create a bathymetry profile from local data files, or by arbitrary creation by the used. This procedure is most easily learned by examining the script file and the comments contained therein.

10 Examples

10.1 1HD: Breaking solitary wave runup and rundown

Solitary wave runup and rundown was investigated experimentally by Synolakis (1986,1987). In his work, dozens of experimental trials were performed, encompassing two orders of magnitude of solitary wave height. The beach slope was kept constant at 1:19.85. Many researchers have used this data set to validate numerical models (e.g., Zelt, 1991; Lin *et al.*, 1999). Synolakis (1986) also photographed the waves during runup and rundown. One set of these snapshots, for $\varepsilon = 0.28$, was digitized and compared with the numerical prediction, shown in Figure 1. The wave begins to break between Figs. 1a) and 1b), and the runup/rundown process is shown in Figs. 1c)-d). In Fig 1c), numerical snapshots from three other models are plotted. The comparisons indicate a significant improvement over weakly nonlinear Boussinesq equation results of Zelt (1991) and the NLSW results of Titov and Synolakis (1995). Additionally, the numerical results by COULWAVE compare favorably to the two dimensional (vertical plane) results of Lin *et al.* (1999), which makes use of a complex turbulence model.

LOCATION OF INPUT FILES: /examples/1D_solitary_wave_up_slope

10.2 1HD: Regular wave evolution over a submerged shoal

The particular case is a good example of the benefits of using the two-layer model over the one-layer (Boussinesq) model. The setup is taken from the experiments presented by Dingemans (1994), who recorded free surface time series at numerous locations in front of and behind the obstacle. The orientation of the bar is shown in the top subplot of Figure 2. The wave, as it approaches the bar, is nearly a long wave, with a kh=0.7 (wavelength of 7.7 m in 0.86 m of water). This incident wave corresponds to Case A in Dingemans (1994). As the wave shoals, it steepens and nonlinear transfers create superharmonics. The superharmonics, while still shallow or intermediate water waves on top of the bar, become deep water waves as they enter the deeper water behind. As discussed in Woo & Liu (2001), significant wave energy (about 75% of the peak spectral amplitude) is present at $kh \approx 4$ in the region behind the bar. For this reason, Boussinesq-type models (one-layer $O(\mu_o^2)$ models), whose linear dispersion accuracy limit is near $kh \approx 3$, do not correctly predict the wave field behind the bar.

Time series are taken at the four locations depicted in the top subplot, and both the one- and two-layer models are compared with experimental data. The column on the left shows the one-layer results, the column on the right, the two-layer. On top of the bar, at location #1, both models are in agreement, and the two-layer model shows no benefit. This is expected, as all of the dominant wave components at this location have kh values less than 2.0. However as the wave components progress into deeper water, the one-layer model becomes inaccurate. This is evident at locations #2-#4, where the one-layer model deviates from the experimental results. The two-layer model, on the other hand, shows its strength and predicts the wave field excellently.

LOCATION OF INPUT FILES: /examples/1D_regular_waves_over_step



Figure 1: Breaking solitary wave runup and rundown on a planar beach at $t(g/h)^{1/2} = a$) 15, b) 20, c) 25, d) 45. The solid line represents the numerical results and the points experimental data. In c) the dashed line represents numerical results by Lin et al. (1999) (closest to experiment and numerical results presented in this paper), the dotted line represents results by Zelt (1991), and the dashed-dotted line results by Titov and Synolakis (1995).



Figure 2: Comparison between numerical (solid lines) and experimental (dots) free surface displacements for Case A of Dingeman (1994), where the experimental setup and gauge locations are shown in the top subplot. The column on the left shows the numerical results from the one-layer model, the right column shows the two-layer results. Time series locations are indicated in the upper right of each subplot, corresponding to the gauge locations shown in the top subplot.



Figure 3: Experimental (dots) and numerical (line) wave height and mean free surface for Hansen and Svendsen case 041041.

10.3 1HD: Breaking regular wave runup and rundown

Hansen and Svendsen (1979) performed a number of regular wave tests on plane slopes. One of these experiments, trail 041041, is recreated numerically by COULWAVE. The waves were generated in 0.36 m of water, and shoaled up a 1:34.26 slope. Time series were taken at numerous locations along the wave flume; wave height and mean free surface elevation will be compared in Figure 3.

LOCATION OF INPUT FILES: /examples/1D_regular_waves_up_slope

10.4 1HD: Breaking irregular wave runup and rundown

Here the identical setup as above is used, except that the regular wave generation is replaced by irregular waves. The input spectrum is generated with the included spectrum_1D.m. A snapshot of the output is provided in Figure 3. Note that this example setup also uses the "Perform spectral analysis" option on the recorded time series. The result of this analysis can be loaded with the included "load_spec.m" file.



Figure 4: TMA spectral waves breaking and running up a plane slope.

LOCATION OF INPUT FILES: /examples/1D_irregular_waves_up_slope

10.5 1HD: Irregular wave overtopping

To demonstrate the models ability to simulate overtopping, this example examines a irregular wave train approaching a bermed levee. The levee profile is created through the user interface (not through external bathymetry files), and the input spectrum is created with the included spectrum_1D.m.

LOCATION OF INPUT FILES: /examples/1D_irregular_waves_overtopping

10.6 2HD: Solitary wave runup and rundown on a conical island

Briggs *et al.* (1994) presented a set of experimental data for solitary wave interaction around a conical island. The slope of the island is 1:4 and the water depth is 0.32m. Three cases were simulated, corresponding to solitary wave heights of $0.013 \ m \ (\varepsilon = 0.04)$, $0.028 \ m \ (\varepsilon = 0.09)$, and $0.058 \ m \ (\varepsilon = 0.18)$. In addition to recording free surface elevation at a half dozen locations, maximum wave runup around the entire island was measured. This data set has been used by several researchers to validate numerical runup models (e.g., Liu *et al.*, 1995; Titov and Synolakis, 1998; Chen *et al.*, 2000). In this paper, free surface elevation is compared at the locations shown in Figure 6. Gages #6 and #9 are located near the front face of the island, with #9 situated very near the initial shoreline position. Gages #16 and #22 are also located at the initial shoreline, where #16 is on the side of the island and #22



Figure 5: TMA spectral waves overtopping a levee with a foreshore berm.

on the back face.

As mentioned, maximum runup was experimentally recorded. The vertical runup heights are converted to horizontal runups, scaled by the initial shoreline radius, and plotted on Figure 7. The crosshairs represents the experimental data, where Fig. 7a) is for Case 1., Fig. 7b) is for Case 2, and Fig. 7c) is for Case 3. The numerical maximum inundation is also plotted, given by the solid line. The agreement for all cases is very good. LOCATION OF INPUT FILES: /examples/2D_solitary_wave_over_shoal

10.7 2HD: Irregular wave evolution over a longshore bar with breaks

This is an example showing how the model can generate wind-induced rip currents and vorticity. A 1D TMA spectrum is generated, and is propagated over a longshore bar system which consists of two breaks. A snapshot of the wave field is given in Figure 9. The return current funnels through these gaps, created large eddies, shown in Figure 9. LOCATION OF INPUT FILES: /examples/2D_irregular_waves_bar_break

10.8 2HD: Directional wave spectrum over nearshore bathymetry

Here, the input wave condition is a directional spectrum, created with the included spectrum_2D. Shown in Figure 10 is snapshot of the wave field, showing the directionality of the



Figure 6: Conical island setup. The gage locations are shown by the dots, and the wave approaches the island from the left.



Figure 7: Maximum horizontal runup, scaled by the initial shoreline radius, for case A a), case B b), and case C c). Experimental values are shown by the stars and the numerical results by the solid line.

incoming wave condition. LOCATION OF INPUT FILES: /examples/2D_irregular_waves_real_beach



Figure 8: Numerical snapshot from the bar-break simulation. The plot is showing the free surface elevation.



Figure 9: Numerical snapshot from the bar-break simulation, showing the formation of eddies and rip currents. The plot is showing the vertical vorticity at the free surface.



Figure 10: Instantaneous snapshot of sea surface elevation for near shore bathymetry simulation.

11 Source Code File List and Description

add_bottom_friction.f - adds bottom friction dissipation to the momentum equations, using the Mannings form. This routine is only called when using the FD scheme. The bottom friction dissipation for the FV scheme is found in FV_efg_calc.f90

add_breaking.f - adds breaking dissipation to the momentum equations

add_sponge.f - adds sponge layer dissipation to the momentum and continuity equations allocate_matrices.f - allocates all the ALLOCATABLE arrays used for the main program bl_define.f - define the outer boundary of the domain

boundary_condition.f - enforces the boundary conditions, both for the moving shoreline and the vertical walls surrounding the domain

calc_Courant_number.f - calculates the maximum local Courant number in the domain; can be used for overflow debugging

calc_corrector_error.f - calculates both the maximum local error and global error; used to determine whether or not the iterative corrector step has converged

calc_flux.f - calculates the mass flux at a (x,y) given location

calc_mass.f - calculates the total mass in the domain; can be used to check for mass conservation

 $calc_maxs_means.f$ - calculates the maximums and mean values of free surface and velocity, for writing to file at the simulation completion

<code>calc_overflow.f</code> - <code>can</code> be used to determine if the simulation has crashed and force the simulation to stop

calc_vel_DA.f - calculates the depth-averaged horizontal velocity components at a given (\mathbf{x},\mathbf{y}) location

calc_vel_z.f - calculates the three dimensional velocity components at a given (x,y,z) location coulwave_MPI.f - the main routine; includes the primary logic for the program

create_depth.f - creates the depth grid for the local process

create_file.f - creates and opens output files for the local process

create_write_xyth.f - creates and writes to file the x, y, and time vectors, and writes to file the local depth grid.

 ${\rm decomp} 2{\rm d.f}$ - ${\rm determines}$ the grid decomposition, or division, locations for a parallel simulation

dhdt_calc.f - determines the transients of the depth; used for landslide simulations

exchange2d.f - exchanges REAL boundary data among subdomains for a parallel simulation exchange2d_int.f - exchanges INTEGER boundary data among subdomains for a parallel simulation

find_dxdydt.f - determines the grid spacing and time step to be used by the simulation find_max_hxx.f - finds the maximum bottom slope and curvature that exists in the domain find_max_zeta.f - finds the maximum free surface elevation that exists in the domain

find_ts_indices.f - finds (i,j) indices for all the time series locations

find_wavelengths.f - determines the wavelength for the various incident wave conditions FV_Riemanns4.f90 - the approximate Riemanns solver used by the FV method

 $\rm FV_allocate_matrices.f$ - allocates all the ALLOCATABLE arrays used for the FV method $\rm FV_efg_calc.f90$ - calculates the right hand side of the continuity and momentum equations

for 2D FV simulations

 $\rm FV_efg_calc_1D.f90$ - calculates the right hand side of the continuity and momentum equations for 1D FV simulations

FV_estime.f90 - calculates cell interface properties, used vy the Riemanns solver

 $FV_interp_xei.f$ - polynomial-based cell interface interpolation routine

FV_interp_xoi.f - polynomial-based cell interface interpolation routine

FV_interp_yei.f - polynomial-based cell interface interpolation routine

FV_interp_yoi.f - polynomial-based cell interface interpolation routine

 ${\rm FV_interpolations.} f90$ - limiter-based cell interface interpolation routine

FV_ix_iy_eval.f - determines the shoreline boundary integer markers for the FV method

 $\rm FV_ldu.f90$ - calculates the upper, diagonal, and lower matrices coefficients required for the tridiagonal system (U to u)

FV_limiters.f90 - the various limiters used in the FV method

 $\mathrm{FV}_\mathrm{var_grp.f}$ - the main logic for the FV method

 $\mathrm{FV}_\mathrm{var}_\mathrm{module.f}$ - variable module used for the FV solution

global_dims.f - finds the global dimensions of the simulation grid, performed prior to the domain decomposition

init_variables.f - a simple routine which zeros a matrix, can be used under certain compilers that do not initialize a matrix.

initial_condition.f - determines the initial free surface and velocity condition

internal_source_type2.f - called when using the internal source to generate waves

load_eta_in.f - called when using an input time series to force a simulation; converts that time series in a format that can be used by the internal source generator

<code>load_inputs.f</code> - <code>loads</code> the input parameters contained in <code>sim_set.dat</code>

mainvar_module.f - variable module used for the main program

move_shoreline.f - routine that moves the shoreline location

set_dispersion_coefs.f - prefactors the linear parts of tridiagonal matrix coefficients; only used by the FD method

set_internal_source_coefs.f - calculates the various coefficients (amplitude and frequency) needed by the internal source generator

set_loop_limits.f - sets start and end limits for the i,j loops

shift_matrices_back.f - shifts the main program matrices back one time step; called at the end of the corrector step

solit_cnoidal_ic.f - calculates the solitary wave analytical solution

spectral_analysis.f - a post-processing routine that performs a spectral (FFT) analysis on recorded time series.

 $\text{store_previous_iter.f}$ - called by the iterative corrector step for the purpose of error calculation

tridiag.f - serial tridiagonal matrix solver

tridiagp.f - parallel tridiagonal matrix solver

user_interface.f - the text-based user interface for COULWAVE

var_grp.f - the main routine for the FD method; calculates the right hand side of the continuity and momentum equations, calls the predictor and corrector, and sets the LDU coefficients write_disp_info.f - a routine that can be used to display information to screen while the simulation is running write_inputs.f - writes the input parameters to sim_set.dat

write_maxs_means.f - writes the max and mean values, as calculated by calc_maxs_means.f to file

write_surfaces.f - writes surfaces of the main variables to file

write_timeseries.f - writes time series data to file

12 Appendix

The appendix consists of a a section detailing the properties of the model equations solved by COULWAVE, a section describing the numerical details such as wave breaking, and lastly a collection of papers describing the development of COULWAVE. The first paper, "A multilayer modeling approach to wave modeling", gives the mathematical details regarding the derivation of the model equations solved by COULWAVE. The second paper, "Modeling wave runup with depth-integrated equations", described the moving boundary algorithm employed by COULWAVE. A third paper, "A numerical study of submarine landslide generated waves and runup", shows how the model can be applied to the problem of waves generated by seafloor disturbances. A fourth paper, "A Depth-Integrated Model for Weakly Dispersive, Turbulent, and Rotational Fluid Flows", describes the weakly rotational derivation, and presented the FV method. The paper references are:

Lynett, P. and Liu, P. L.-F. (2004). "A Two-Layer Approach to Water Wave Modeling." Proc. Royal Society of London A. v. 460, p. 2637-2669.

Lynett, P., Wu, T.-R., and Liu, P. L.-F. (2002). "Modeling Wave Runup with Depth-Integrated Equations." Coastal Engineering, v. 46(2), p. 89-107.

Lynett, P. and Liu, P. L.-F. (2002). "A Numerical Study of Submarine Landslide Generated Waves and Runup." Proc. Royal Society of London A. v. 458, p. 2885-2910

Kim, D.-H., Lynett, P. and Socolofosky, S. (2008). "A Depth-Integrated Model for Weakly Dispersive, Turbulent, and Rotational Fluid Flows." submitted to Ocean Modeling.

13 Appendix A. Analysis of Multi-Layer Models

13.1 One-Layer Equation Model

For the one-layer model, the horizontal velocity vector is given as

$$\boldsymbol{U}_{1} = \boldsymbol{u}_{1} - \mu_{o}^{2} \left\{ \frac{z_{1}^{2} - \kappa_{1}^{2}}{2} \nabla S_{1} + (z_{1} - \kappa_{1}) \nabla T_{1} \right\} + O(\mu_{o}^{4})$$
(2)

where

$$S_1 = \nabla \cdot \boldsymbol{u}_1, \qquad T_1 = \nabla \cdot (h\boldsymbol{u}_1) + \frac{1}{\varepsilon_o} \frac{\partial h}{\partial t}$$
 (3)

The exact continuity equation can be rewritten approximately in terms of ζ and u_1 as:

$$\frac{1}{\varepsilon_o} \frac{\partial h}{\partial t} + \frac{\partial \zeta}{\partial t} + \nabla \cdot \left[(\varepsilon_o \zeta + h) \, \boldsymbol{u}_1 \right] -\mu_o^2 \nabla \cdot \left\{ \left[\frac{\varepsilon_o^3 \zeta^3 + h^3}{6} - \frac{(\varepsilon_o \zeta + h) \kappa_1^2}{2} \right] \nabla S_1 + \left[\frac{\varepsilon_o^2 \zeta^2 - h^2}{2} - (\varepsilon_o \zeta + h) \kappa_1 \right] \nabla T_1 \right\} = O(\mu_o^4)$$
(4)

Equation (4) is one of two governing equations for ζ and u_1 . The momentum equation for u_1 is

$$\frac{\partial \boldsymbol{u}_{1}}{\partial t} + \varepsilon_{o}\boldsymbol{u}_{1} \cdot \nabla \boldsymbol{u}_{1} + \nabla \zeta + \mu_{o}^{2} \frac{\partial}{\partial t} \left\{ \frac{\kappa_{1}^{2}}{2} \nabla S_{1} + \kappa_{1} \nabla T_{1} \right\} \\
+ \varepsilon_{o}\mu_{o}^{2} \left[(\boldsymbol{u}_{1} \cdot \nabla \kappa_{1}) \nabla T_{1} + \kappa_{1} \nabla (\boldsymbol{u}_{1} \cdot \nabla T_{1}) + \kappa_{1} (\boldsymbol{u}_{1} \cdot \nabla \kappa_{1}) \nabla S_{1} + \frac{\kappa_{1}^{2}}{2} \nabla (\boldsymbol{u}_{1} \cdot \nabla S_{1}) \right] \\
+ \varepsilon_{o}\mu_{o}^{2} \left[T_{1} \nabla T_{1} - \nabla \left(\zeta \frac{\partial T_{1}}{\partial t} \right) \right] + \varepsilon_{o}^{2}\mu_{o}^{2} \nabla \left(\zeta S_{1}T_{1} - \frac{\zeta^{2}}{2} \frac{\partial S_{1}}{\partial t} - \zeta \boldsymbol{u}_{1} \cdot \nabla T_{1} \right) \\
+ \varepsilon_{o}^{3}\mu_{o}^{2} \nabla \left[\frac{\zeta^{2}}{2} \left(S_{1}^{2} - \boldsymbol{u}_{1} \cdot \nabla S_{1} \right) \right] = O(\mu_{o}^{4})$$
(5)

This one-layer model, often referred to as the "fully nonlinear, extended Boussinesq equations" in the literature (e.g. Wei & Kirby, 1995), has been examined and applied to a significant extent. The weakly nonlinear version of (4) and (5) (i.e. assuming $O(\varepsilon_o) = O(\mu_o^2)$, thereby neglecting all nonlinear dispersive terms) was first derived by Nwogu (1993). Nwogu, through linear and first-order nonlinear analysis of the equation model, recommended that $z_1 = -0.531h$, and that value has been, for the most part, adopted by other researchers using these equations. Nwogu's model was extended to "full nonlinearity" by Liu (1994) and Wei & Kirby (1995). There are some discrepancies between Liu's and Wei & Kirby's derived equations, which can be attributed to a neglect of some nonlinear dispersive terms in Wei & Kirby (Hsaio & Liu, 2002). The above, one-layer model equations (4) and (5) are identical to those derived by Liu (1994).

The one-layer model has been used to study a number of 2HD real world phenomenon, including rip currents (Chen et al., 1999), longshore currents (Chen et al., 2002), and a variety of harbor problems (e.g. Shi et al., 2002). The numerical scheme employed for these simulations is adopted here for the two-layer model, and will be described in detail in Chapter 4.

13.2 Two-Layer Equation Model

For the two-layer model, we can define the horizontal velocity vectors as

$$\boldsymbol{U}_{2} = \boldsymbol{u}_{2} - \mu_{2}^{2} \left\{ \frac{z_{2}^{2} - \kappa_{2}^{2}}{2} \nabla S_{2} + (z_{2} - \kappa_{2}) \nabla T_{2} \right\} + O(\mu_{2}^{4})$$
(6)

$$\boldsymbol{U}_{1} = \boldsymbol{u}_{1} - \mu_{1}^{2} \left\{ \frac{z_{1}^{2} - \kappa_{1}^{2}}{2} \nabla S_{1} + (z_{1} - \kappa_{1}) \nabla T_{1} \right\} + O(\mu_{1}^{4}, \mu_{1}^{2} \mu_{2}^{2})$$
(7)

where

$$S_{2} = \frac{d_{2}}{h_{o}} \nabla \cdot \boldsymbol{u}_{2}, \qquad T_{2} = \nabla \cdot (h\boldsymbol{u}_{2}) + \frac{1}{\varepsilon_{o}} \frac{\partial h}{\partial t}$$
$$S_{1} = \frac{d_{1}}{h_{o}} \nabla \cdot \boldsymbol{u}_{1}, \qquad T_{1} = \eta \left(\frac{d_{1}}{d_{2}}S_{2} - S_{1}\right) + T_{2} \qquad (8)$$

The exact continuity equation can be rewritten approximately in terms of ζ , u_1 , and u_2 as:

$$\frac{h_o}{d_1\varepsilon_o}\frac{\partial h}{\partial t} + \frac{h_o}{d_1}\frac{\partial \zeta}{\partial t} + \nabla \cdot \left[(\varepsilon_1\zeta - \eta) \, \boldsymbol{u}_1 + \left(\eta + \frac{h_o}{d_1}h\right) \, \boldsymbol{u}_2 \right]
-\mu_2^2 \frac{d_2}{d_1} \nabla \cdot \left\{ \left[\frac{\eta^3 \frac{d_1^3}{d_2^3} + h^3 \frac{h_o^3}{d_2^3}}{6} - \frac{(\eta \frac{d_1}{d_2} + h \frac{h_o}{d_2})\kappa_2^2}{2} \right] \nabla S_2
+ \left[\frac{\eta^2 \frac{d_1^2}{d_2^2} - h^2 \frac{h_o^2}{d_2^2}}{2} - (\eta \frac{d_1}{d_2} + h \frac{h_o}{d_2})\kappa_2 \right] \nabla T_2 \right\}
-\mu_1^2 \nabla \cdot \left\{ \left[\frac{\varepsilon_1^3 \zeta^3 - \eta^3}{6} - \frac{(\varepsilon_1 \zeta - \eta)\kappa_1^2}{2} \right] \nabla S_1
+ \left[\frac{\varepsilon_1^2 \zeta^2 - \eta^2}{2} - (\varepsilon_1 \zeta - \eta)\kappa_1 \right] \nabla T_1 \right\} = O(\mu_1^4, \mu_1^2 \mu_2^2, \mu_2^4)$$
(9)

Equation (9) is one of three governing equations for ζ and \boldsymbol{u}_n . The governing, momentum equation for \boldsymbol{u}_1 is

$$\frac{\partial \boldsymbol{u}_{1}}{\partial t} + \varepsilon_{o}\boldsymbol{u}_{1} \cdot \nabla \boldsymbol{u}_{1} + \nabla \zeta + \mu_{1}^{2} \frac{\partial}{\partial t} \left\{ \frac{\kappa_{1}^{2}}{2} \nabla S_{1} + \kappa_{1} \nabla T_{1} \right\}$$

$$+ \varepsilon_{o}\mu_{1}^{2} \left[(\boldsymbol{u}_{1} \cdot \nabla \kappa_{1}) \nabla T_{1} + \kappa_{1} \nabla (\boldsymbol{u}_{1} \cdot \nabla T_{1}) + \kappa_{1} (\boldsymbol{u}_{1} \cdot \nabla \kappa_{1}) \nabla S_{1} + \frac{\kappa_{1}^{2}}{2} \nabla (\boldsymbol{u}_{1} \cdot \nabla S_{1}) \right]$$

$$+ \varepsilon_{o}\mu_{o}^{2} \left[T_{1} \nabla T_{1} - \nabla \left(\zeta \frac{\partial T_{1}}{\partial t} \right) \right] + \varepsilon_{o}^{2}\mu_{o}^{2} \nabla \left(\zeta S_{1}T_{1} - \frac{h_{o}}{d_{1}} \frac{\zeta^{2}}{2} \frac{\partial S_{1}}{\partial t} - \zeta \boldsymbol{u}_{1} \cdot \nabla T_{1} \right)$$

$$+ \varepsilon_{o}^{2}\varepsilon_{1}\mu_{o}^{2} \nabla \left[\frac{\zeta^{2}}{2} \left(S_{1}^{2} - \frac{h_{o}}{d_{1}} \boldsymbol{u}_{1} \cdot \nabla S_{1} \right) \right] = O(\mu_{1}^{4}, \mu_{0}^{2}\mu_{1}^{2}, \mu_{1}^{2}\mu_{2}^{2}) \tag{10}$$

Determination of u_2 does not require solving an additional momentum equation. With interfacial boundary condition (continuous velocity) and the known velocity profiles (6) and (7), u_2 can be explicitly given as a function of u_1 :

$$\boldsymbol{u}_{2} + \mu_{2}^{2} \left\{ \frac{\kappa_{2}^{2} - \frac{d_{1}^{2}}{d_{2}^{2}} \eta^{2}}{2} \nabla S_{2} + \left(\kappa_{2} - \frac{d_{1}}{d_{2}} \eta\right) \nabla T_{2} \right\} =$$
$$\boldsymbol{u}_{1} + \mu_{1}^{2} \left\{ \frac{\kappa_{1}^{2} - \eta^{2}}{2} \nabla S_{1} + (\kappa_{1} - \eta) \nabla T_{1} \right\} + O(\mu_{1}^{4}, \mu_{1}^{2} \mu_{2}^{2}, \mu_{2}^{4})$$
(11)

Thus, the lower layer velocity can be directly calculated with knowledge of the upper layer velocity. Equations (9), (10), and (11) are the coupled governing equations for the two-layer system.

13.2.1 Analysis of Model Equations

In this section, the properties of the two-layer model will be scrutinized and optimized. First, it is shown that the two-layer model will reduce to the well-studied, "extended" Boussinesq model derived by Ngowu (1993). With the use of $O(\mu_n^2)$ substitutions, namely:

$$\boldsymbol{u}_2 = \boldsymbol{u}_1 + O(\mu_n^2), \tag{12}$$

we can eliminate one of the unknowns from our equation system. Rewriting (9) in terms of \boldsymbol{u}_1 only, assigning $d_1 = h_o$, $\kappa_2 = -\frac{h_o}{d_2}h$, $\eta = -h$, and examining the weakly nonlinear form of the equations, gives

$$\frac{1}{\varepsilon_o} \frac{\partial h}{\partial t} + \frac{\partial \zeta}{\partial t} + \nabla \cdot \left[\left(\varepsilon_o \zeta + h \right) \boldsymbol{u}_1 \right] -\mu_o^2 \nabla \cdot \left\{ \left[\frac{h^3}{6} - \frac{h\kappa_1^2}{2} \right] \nabla S_1^* - \left[\frac{h^2}{2} + h\kappa_1 \right] \nabla T_1^* \right\} = O(\varepsilon_o \mu_o^2, \ \mu_o^4)$$
(13)

where

$$S_1^* = \nabla \boldsymbol{u}_1, \quad T_1^* = \nabla \cdot (h\boldsymbol{u}_1) + \frac{1}{\varepsilon_o} \frac{\partial h}{\partial t}$$
 (14)

The momentum equation, (10), becomes

$$\frac{\partial \boldsymbol{u}_1}{\partial t} + \varepsilon_o \boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1 + \nabla \zeta + \mu_o^2 \frac{\partial}{\partial t} \left\{ \frac{\kappa_1^2}{2} \nabla S_1^* + \kappa_1 \nabla T_1^* \right\} = O(\varepsilon_o \mu_o^2, \mu_o^4)$$
(15)

This system for ζ and u_1 is identical to the model derived by Ngowu. Additionally, the nonlinear dispersive terms, which have been truncated for the sake of brevity in (13) and (15), are identical to those derived by Liu (1994). For the rest of this paper, the "extended" Boussinesq model including all the nonlinear dispersive terms up to $O(\mu_o^2)$, as given by Liu (1994), will be referred to as the one-layer model.

For the rest of this section, the focus will be on analysis of the three-unknown, $(\zeta, u_1, and u_2)$ two-layer system. Additionally for the rest of this section, all quantities discussed are in dimensional form, with asterisks no longer applied. With the weak rotationality assumption,

the momentum equation, (10), can be simplified, in dimensional form, to (see Hsiao and Liu, 2002)

$$\frac{\partial \boldsymbol{u}_{1}}{\partial t} + \frac{1}{2}\nabla(\boldsymbol{u}_{1}\cdot\boldsymbol{u}_{1}) + g\nabla\zeta + \frac{\partial}{\partial t}\left\{\frac{\kappa_{1}^{2}}{2}\nabla S_{1} + \kappa_{1}\nabla T_{1} - \nabla\left(\frac{\zeta^{2}}{2}S_{1}\right) - \nabla\left(\zeta T_{1}\right)\right\} \\
+ \nabla\left\{\frac{\partial\zeta}{\partial t}\left(T_{1} + \zeta S_{1}\right) + \left(\kappa_{1} - \zeta\right)\left(\boldsymbol{u}_{1}\cdot\nabla\right)T_{1} + \frac{1}{2}\left(\kappa_{1}^{2} - \zeta^{2}\right)\left(\boldsymbol{u}_{1}\cdot\nabla\right)S_{1} \\
+ \frac{1}{2}\left[\left(T_{1} + \zeta S_{1}\right)^{2}\right]\right\} = 0$$
(16)

This is the momentum equation that will be analyzed and numerically solved in this paper. Before solving the system, the linear and nonlinear dispersion properties are examined. Let us define the arbitrary evaluation levels and the boundary between the two layers as:

$$\kappa_1 = \alpha_1 h + \beta_1 \zeta, \quad \eta = \alpha_2 h + \beta_2 \zeta, \quad \kappa_2 = \alpha_3 h + \beta_3 \zeta$$
(17)

where the coefficients α and β are arbitrary and user-defined. The one-horizontal dimension, constant water depth, two-layer equations are rewritten in dimensional form, keeping, for brevity, only linear terms. These equations are

$$\frac{\partial \zeta}{\partial t} + \delta_1 h \frac{\partial \boldsymbol{u}_1}{\partial x} + \delta_2 h \frac{\partial \boldsymbol{u}_2}{\partial x} + \delta_3 h^3 \frac{\partial^3 \boldsymbol{u}_1}{\partial x^3} + \delta_4 h^3 \frac{\partial^3 \boldsymbol{u}_2}{\partial x^3} = 0$$
(18)

$$\frac{\partial \boldsymbol{u}_1}{\partial t} + g \frac{\partial \zeta}{\partial x} + \delta_5 h^2 \frac{\partial^3 \boldsymbol{u}_1}{\partial x^2 t} + \delta_6 h^2 \frac{\partial^3 \boldsymbol{u}_2}{\partial x^2 t} = 0$$
(19)

$$\boldsymbol{u}_1 - \boldsymbol{u}_2 - \delta_7 h^2 \frac{\partial^2 \boldsymbol{u}_1}{\partial x^2} - \delta_8 h^2 \frac{\partial^2 \boldsymbol{u}_2}{\partial x^2} = 0$$
(20)

where

$$\delta_{1} = -\alpha_{2}, \qquad \delta_{2} = 1 + \alpha_{2}, \qquad \delta_{3} = \frac{-2\alpha_{2}^{3} + 6\alpha_{1}\alpha_{2}^{2} - 3\alpha_{1}^{2}\alpha_{2}}{6},$$

$$\delta_{4} = \frac{2\alpha_{2}^{3} - 6\alpha_{1}\alpha_{2}^{2} - 6\alpha_{1}\alpha_{2} + 3\alpha_{3}^{2}\alpha_{2} + 6\alpha_{3}\alpha_{2} + 3\alpha_{3}^{2} + 6\alpha_{3} + 2}{6},$$

$$\delta_{5} = \frac{\alpha_{1}^{2}}{2} - \alpha_{1}\alpha_{2}, \qquad \delta_{6} = \alpha_{1}\alpha_{2} + \alpha_{1}, \qquad \delta_{7} = -\frac{\alpha_{1}^{2} + \alpha_{2}^{2}}{2} + \alpha_{1}\alpha_{2}$$

$$\delta_{8} = \frac{\alpha_{2}^{2} + \alpha_{3}^{2}}{2} - \alpha_{1}\alpha_{2} + \alpha_{3} - \alpha_{1} \qquad (21)$$

The assumed dimensional solution form

$$\zeta = \epsilon \zeta^{(0)} e^{i\theta} + \epsilon^2 \zeta^{(1)} e^{2i\theta} + \dots$$

$$\boldsymbol{u}_1 = \epsilon \boldsymbol{u}_1^{(0)} e^{i\theta} + \epsilon^2 \boldsymbol{u}_1^{(1)} e^{2i\theta} + \dots \qquad \boldsymbol{u}_2 = \epsilon \boldsymbol{u}_2^{(0)} e^{i\theta} + \epsilon^2 \boldsymbol{u}_2^{(1)} e^{2i\theta} + \dots \qquad (22)$$

where $\theta = kx - wt$, k is the wavenumber, w is the wave frequency, and ϵ is simply an ordering parameter, are substituted into the derived equations.

13.2.2 Linear Dispersion Relation

The first order (in ϵ) system yields the linear dispersion relation:

$$c^{2} = \frac{w^{2}}{k^{2}} = \frac{gh\left[1 + N_{1}(kh)^{2} + N_{2}(kh)^{4}\right]}{1 + D_{1}(kh)^{2} + D_{2}(kh)^{4}}$$
(23)

where c is the wave celerity and the coefficients N_1 , N_2 , D_1 , and D_2 are given in Appendix A.1 and are solely functions of α_1 , α_2 , and α_3 . The above dispersion relation will be compared with both the [4,4] Pade approximation

$$c^{2} = \frac{w^{2}}{k^{2}} = \frac{gh\left[1 + \frac{1}{9}(kh)^{2} + \frac{1}{9}45(kh)^{4}\right]}{1 + \frac{4}{9}(kh)^{2} + \frac{1}{63}(kh)^{4}}$$
(24)

and the [6,6] Pade approximation

$$c^{2} = \frac{w^{2}}{k^{2}} = \frac{gh \left[1 + \frac{5}{39(kh)^{2}} + \frac{2}{715(kh)^{4}} + \frac{1}{135135(kh)^{6}}\right]}{1 + \frac{6}{13(kh)^{2}} + \frac{10}{429(kh)^{4}} + \frac{4}{19305(kh)^{6}}}$$
(25)

of the exact linear dispersion relation:

$$c_e^2 = \frac{w^2}{k^2} = \frac{g}{k} tanh(kh) \tag{26}$$

The Pade approximates utilized here are approximations of the hyperbolic tangent function, where the numbers in the brackets represent the highest polynomial order of kh in the numerator and denominator. Group velocity of the two-layer model equations, c_g , can be determined straightforwardly by taking the derivative of (23) with respect to k.

13.2.3 Vertical Velocity Profile

Let us define the function $f_1(z)$ as the horizontal velocity, with constant water depth, normalized by its value at z = 0. This function is composed of two quadratic polynomial elements, given by:

$$f_1(z) = \frac{1 + (kh)^2 \left[\frac{1}{2} \left(\frac{z^2}{h^2} - \alpha_1^2\right) + \alpha_2(\alpha_1 - z/h) + u_2^{(0)} / u_1^{(0)}(\alpha_2 + 1)(z/h - \alpha_1)\right]}{1 - (kh)^2 \left[\frac{1}{2}\alpha_1^2 - \alpha_2\alpha_1 + u_2^{(0)} / u_1^{(0)}(\alpha_2 + 1)\alpha_1\right]},$$

for
$$z \ge \eta = \alpha_2 h$$
 (27)

$$f_1(z) = f_1(\eta) \frac{1 + (kh)^2 \left[\frac{1}{2} \left(\frac{z^2}{h^2} - \alpha_3^2\right) + \left(\frac{z}{h} - \alpha_3\right)\right]}{1 + (kh)^2 \left[\frac{1}{2} \left(\alpha_2^2 - \alpha_3^2\right) + \left(\alpha_2 - \alpha_3\right)\right]}, \quad \text{for } z < \eta = \alpha_2 h$$
(28)

From the linear equation system we know that,

$$u_1^{(0)} = \frac{g\zeta^{(0)} \left[kh - \delta_8(kh)^3\right]}{hw \left[1 + D_1(kh)^2 + D_2(kh)^4\right]}$$
(29)

$$u_2^{(0)} = \frac{g\zeta^{(0)} \left[kh + \delta_7 (kh)^3\right]}{hw \left[1 + D_1 (kh)^2 + D_2 (kh)^4\right]}$$
(30)

and thus the ratio $u_2^{(0)}/u_1^{(0)}$ present in (27) can be evaluated. Similarly, the vertical velocity profile, normalized by the velocity at the still water level, is given by $f_2(z)$:

$$f_2(z) = \frac{z/h - \alpha_2 + u_2^{(0)}/u_1^{(0)}(\alpha_2 + 1)}{-\alpha_2 + u_2^{(0)}/u_1^{(0)}(\alpha_2 + 1)}, \quad \text{for } z \ge \eta = \alpha_2 h$$
(31)

$$f_2(z) = f_2(\eta) \frac{z/h+1}{\alpha_2+1}, \text{ for } z < \eta = \alpha_2 h$$
 (32)

which is a piecewise linear function.

13.2.4 Linear Shoaling Properties

Based on linear theory, the exact shoaling gradient is given as:

$$\frac{a_x^e}{a} = A_x^e \frac{h_x}{h} = -kh \tanh(kh) \frac{[1 - kh \tanh(kh)][1 - \tanh^2(kh)]}{\{\tanh(kh) + kh[1 - \tanh^2(kh)]\}^2} \frac{h_x}{h}$$
(33)

The linear shoaling properties of the two layer model are determined using the constancy of energy flux concept, i.e

$$\frac{a_x}{a} = -\frac{1}{2} \frac{(C_g)_x}{C_g}$$
(34)

where C_g is the wave group velocity. First, the derivative of (23) is taken with respect to k, giving:

$$\frac{w}{g}c_g = \frac{(kh)S_1}{S_2^2} \tag{35}$$

where c_g is the wave group velocity, and

$$S_1 = D_2 N_2 (kh)^8 + 2D_1 N_2 (kh)^6 + (3N_2 + D_1 N_1 - D_2) (kh)^4 + 2N_1 (kh)^2 + 1$$
(36)

$$S_2 = D_2(kh)^4 + D_1(kh)^2 + 1$$
(37)

Taking the derivative of (35) with respect to x, noting that dw/dx=0, we have

$$\frac{w}{g}(c_g)_x = (kh)_x \frac{S_3}{S_2^3}$$
(38)

where

$$S_{3} = D_{2}^{2}N_{2}(kh)^{12} + 3D_{1}D_{2}N_{2}(kh)^{10} + (6D_{1}^{2}N_{2} - 3D_{1}D_{2}N_{1} + 3D_{2}^{2})(kh)^{8} + (17D_{1}N_{2} - 10D_{2}N_{1} + D_{1}^{2}N_{1} - D_{1}D_{2})(kh)^{6} + (15N_{2} + 3D_{1}N_{1} - 12D_{2})(kh)^{4} + (6N_{1} - 3D_{1})(kh)^{2} + 1$$
(39)

giving the ratio

$$\frac{(c_g)_x}{c_g} = \frac{(kh)_x}{kh} \frac{S_3}{S_1 S_2}$$
(40)

Taking the derivative of the dispersion relation (23), with respect to x, gives

$$\frac{k_x}{k} = -\frac{1}{2} \frac{S_4}{S_1} \frac{h_x}{h}$$
(41)

where

$$S_4 = D_2 N_2 (kh)^8 + (3D_1 N_2 - D_2 N_1) (kh)^6 + (5N_2 + D_1 N_1 - 3D_2) (kh)^4 + (3N_1 - D_1) (kh)^2 + 1$$
(42)

Finally, the linear shoaling gradient of the two-layer model can be given:

$$\frac{a_x}{a} = -\left(\frac{1}{2} - \frac{S_4}{4S_1}\right) \frac{S_3}{S_1 S_2} \frac{h_x}{h}$$
(43)

Note that this solution form is valid in any system for which the dispersion relation can be expressed in the form of (23).

13.2.5 Second Order, Nonlinear Interactions: Steady Waves

Now we find the nonlinear corrections to the linear problem. The two-layer equations must now be truncated to include quadratic nonlinear terms, as well as linear terms. Collecting the $O(\epsilon^2)$ terms from the substitution of the assumed steady wave, (22), into the nonlinear equation system will yield an equation system in the general form:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix} \begin{pmatrix} \zeta^{(1)} \\ \boldsymbol{u}_1^{(1)} \\ \boldsymbol{u}_2^{(1)} \end{pmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

where $b_{11}, ..., b_{33}$ are functions of the linear δ coefficients, and $R_1, ..., R_3$ are tedious functions of the α and β parameters. This approximate expression can be compared to the secondorder solution:

$$\zeta_{Stokes}^{(1)} = \frac{k\zeta^{(0)^2}}{4} [3\coth^3(kh) - \coth(kh)]$$
(44)

which is derived from Stokes theory.

13.2.6 Second Order, Nonlinear Interactions: Bichromatic Interactions

Examining a two-wave group, the free surface can be written as

$$\zeta = \epsilon \zeta_1^{(0)} e^{i(k_1 x - w_1 t)} + \epsilon \zeta_2^{(0)} e^{i(k_2 x - w_2 t)} + \epsilon^2 \zeta_1^{(1)} e^{2i(k_1 x - w_1 t)} + \epsilon^2 \zeta_2^{(1)} e^{2i(k_2 x - w_2 t)} + \epsilon^2 \zeta^+ e^{i(k_1 x - w_1 t)} + \epsilon^2 \zeta^- e^{i(k_1 x - w_1 t)}$$
(45)

where ζ_+, ζ_- are the sum and difference components of the two first order wave frequencies, $k_{\mp} = k_1 \mp k_2$, and $w_{\mp} = w_1 \mp w_2$. Similar expressions can be given for u_n . To find the sub- and super-harmonic amplitudes for the bichromatic wave group problem, the procedure is the same as described above for the steady wave (single, first-order harmonic) problem. The assumed solution (45) is substitued into the two-layer equation system. For each of the forced second-order solutions, $[(k_1 - k_2)x - (w_1 - w_2)t]$ and $[(k_1 + k_2)x + (w_1 - w_2)t]$, the matrix system is written in the same form as for the steady wave problem. The sum and difference free surface components can be compared with those from Stokes theory, ζ_{Stokes}^{\mp} , which can be found in Shaffer (1996).

		-		v
$\Omega~({ m kh})$	α_1	α_2	$lpha_3$	Δ_{Linear}
3	-0.225	-0.420	-0.713	0.00008
5	-0.204	-0.383	-0.685	0.002
7.5	-0.175	-0.331	-0.646	0.009
10	-0.155	-0.294	-0.620	0.020
[4,4] Pade :	-0.248	-0.459	-0.741	

Table 1: α values from linear optimization for two-layer model.

13.2.7 Choice of Arbitrary Levels: Linear Optimization

Through examination of linear and nonlinear properties, the most accurate set of arbitrary levels will be chosen in this section. First, the linear properties of the two-layer model will be optimized, independent of nonlinearity. In the linear sense, the three levels are given as $\kappa_1 = \alpha_1 h$, $\eta = \alpha_2 h$, and $\kappa_2 = \alpha_3 h$, where κ_1 and κ_2 are the levels at which horizontal velocities are evaluated in the upper and lower layers, and η is the location of the interface between the layers. Of course, possible values are bounded by $0 \ge \alpha_1 \ge \alpha_2 \ge \alpha_3 \ge -1$. Defining a model accuracy, or model error, can be difficult and often can depend on the specific physical problem being examined. For this analysis, a representation of the overall error, including errors in wave speed, group velocity, and shoaling, is sought. The error will be given by the minimization parameter Δ_{Linear} :

$$\Delta_{Linear} = \frac{1}{3} \left(\frac{\sum_{kh=0.1}^{\Omega} \frac{|c^e - c|}{kh}}{\sum_{kh=0.1}^{\Omega} \frac{|c^e|}{kh}} + \frac{\sum_{kh=0.1}^{\Omega} \frac{|c^e_g - c_g|}{kh}}{\sum_{kh=0.1}^{\Omega} \frac{|c^e_g|}{kh}} + \frac{\sum_{kh=0.1}^{\Omega} \frac{|A^e_x - A_x|}{kh}}{\sum_{kh=0.1}^{\Omega} \frac{|A^e_x|}{kh}} \right)$$
(46)

where c^e , c_g^e , and A_x^e are the exact linear phase speed, group velocity, and shoaling gradient, whereas c, c_g , and A_x are the approximate values taken from the two-layer model derived here. The right hand side is divided by three, so as to normalize the total error created by the three different sources. All of the summations are divided by kh so that errors at low wave numbers are more important than high wave number errors. The reason for this weighting is a peculiarity of the optimization: it was possible to sacrifice low wavenumber accuracy (kh < 1.5) for accuracy at higher wavenumbers. Accuracy at low wavenumbers is paramount, and hence the weighting. Summations are started at kh = 0.1 also because of the kh weighting, and the subsequent need to avoid division by zero. The upper summation limit, $kh = \Omega$, is determined such that the minimum Δ_{Linear} is less than some threshold.

 Δ_{Linear} , which can be thought of as an overall relative error, will be set equal to four arbitrary values. The behavior of the equation model at these error constraints will be scrutinized, and a "proper" Δ_{Linear} value will be recommended. A summary of the optimization results is shown in Table 1. Also shown in last row of the table are the α values required to create a [4,4] Pade approximation using the two-layer dispersion relation. Figures 11 -14 show the linear properties for the cases given in Table 1. All of these figures also show the dispersion properties corresponding to the [4,4] Pade. The [4,4] Pade yields excellent phase speed agreement up to kh values of 6, good group velocity agreement to kh near 3, and an accurate shoaling gradient to a kh of 2. For the $\Delta_{Linear}=0.002$ case, we can see that the linear dispersion properties (phase and group speed) lye inbetween a [4,4] and a [6,6] Pade approximation. Linear shoaling is reproduced very well up to kh=5. Note that the optimized α_2 value for this case is the same as the value derived previously from the heuristic analysis in section 4. Looking now at the $\Delta_{Linear}=0.020$ case, the model phase speed has better deep-water accuracy than a [6,6] Pade approximation. The price paid for this increased accuracy is a group velocity that oscillates around the exact linear group velocity with an error amplitude of 1-2%, with the error starting near kh=1. Additionally, the shoaling gradient diverges slightly from the exact solution at lower kh than the $\Delta=0.002$ case, although the agreement is still reasonable for kh values to 8.

For the rest of the paper, all the results will employ the α values from the $\Delta_{Linear}=0.002$ minimization. This set was chosen based on its middle-of-the-road overall properties. It can be expected that phase and group velocity will be well captured for kh values up to 8, and linear shoaling will be excellent up to a kh of 5. It was decided that the $\Delta_{Linear}=0.020$ optimization was unacceptable due to the small, but low kh, errors in the group velocity prediction. For long channel, wave group simulations, a 2% error in group velocity will accumulate in time, eventually destroying the accuracy of a simulation. Practically, however, it may be reasonable to employ the $\Delta_{Linear}=0.020$ optimization coefficients, depending on the specifics of the problem.

The vertical velocity profiles predicted with the four different Δ_{Linear} values are given in Figures 15-18. Also plotted on these figures are the velocity profiles of Gobbi *et al.*'s (2000) high-order derivation, which is a one-layer model, including terms up to $O(\mu_o^4)$. The pattern of error in the velocity profiles follows very closely to that shown in the phase velocity comparisons. For the $\Delta_{Linear}=0.00008$ case, in Fig. 15, the kh=3 profile shows extremely good agreement with linear theory, however as kh increases the agreement drops off. On the opposite end for the $\Delta_{Linear}=0.02$ case, in Fig. 18, the velocity profile agreement is very good even to kh=9. The cost of this high wavenumber accuracy is error at kh=3. This high kh accuracy/ low kh error tradeoff is identical to what is seen with the phase velocity comparisons.

13.2.8 Choice of Arbitrary Levels: Nonlinear Optimization

From the linear optimization of the previous section, the three levels can be given as:

$$\kappa_1 = -0.204h + \beta_1 \zeta, \quad \eta = -0.383h + \beta_2 \zeta, \quad \kappa_2 = -0.685h + \beta_3 \zeta$$
(47)

In this section, through examination of nonlinear properties, the β coefficients will be chosen. The nonlinear optimization detailed in this section is similar to that performed by Kennedy *et al.* (2001) while working with the one-layer model. Following the same procedure as the linear optimization, a representation of the nonlinear error, including errors in the second order free surface correction and subharmonic/superharmonic transfer functions is given by



Figure 11: Properties of two-layer model with $\alpha_1 = -0.225$, $\alpha_2 = -0.420$, and $\alpha_3 = -0.713$ ($\Delta_{Linear}=0.00008$). Comparison of wave speed and group velocity of the two-layer model (dashed line) with the exact linear relation (solid line); the dotted line is the [4,4] Pade, and the dashed-dotted line is the [6,6] Pade. The linear shoaling factor is shown in c), where the [6,6] Pade is not shown.


Figure 12: Properties of two-layer model with $\alpha_1 = -0.204$, $\alpha_2 = -0.383$, and $\alpha_3 = -0.685$ ($\Delta_{Linear}=0.002$). Comparison of wave speed and group velocity of the two-layer model (dashed line) with the exact linear relation (solid line); the dotted line is the [4,4] Pade, and the dashed-dotted line is the [6,6] Pade. The linear shoaling factor is shown in c), where the [6,6] Pade is not shown.



Figure 13: Properties of two-layer model with $\alpha_1 = -0.175$, $\alpha_2 = -0.331$, and $\alpha_3 = -0.646$ ($\Delta_{Linear}=0.009$). Comparison of wave speed and group velocity of the two-layer model (dashed line) with the exact linear relation (solid line); the dotted line is the [4,4] Pade, and the dashed-dotted line is the [6,6] Pade. The linear shoaling factor is shown in c), where the [6,6] Pade is not shown.



Figure 14: Properties of two-layer model with $\alpha_1 = -0.155$, $\alpha_2 = -0.294$, and $\alpha_3 = -0.620$ ($\Delta_{Linear}=0.020$). Comparison of wave speed and group velocity of the two-layer model (dashed line) with the exact linear relation (solid line); the dotted line is the [4,4] Pade, and the dashed-dotted line is the [6,6] Pade. The linear shoaling factor is shown in c), where the [6,6] Pade is not shown.



Figure 15: Vertical profile of horizontal velocity (top row) and vertical velocity (bottom row) under the crest of a sine wave for three different kh values, as given by linear theory (solid line), the high-order model of Gobbi *et al.* (2000) (dotted line), and the 2-layer model presented in this paper employing the $\Delta_{Linear}=0.00008$ coefficients (dashed line).



Figure 16: Vertical profile of horizontal velocity (top row) and vertical velocity (bottom row) under the crest of a sine wave for three different kh values, as given by linear theory (solid line), the high-order model of Gobbi *et al.* (2000) (dotted line), and the 2-layer model presented in this paper employing the $\Delta_{Linear}=0.002$ coefficients (dashed line).



Figure 17: Vertical profile of horizontal velocity (top row) and vertical velocity (bottom row) under the crest of a sine wave for three different kh values, as given by linear theory (solid line), the high-order model of Gobbi *et al.* (2000) (dotted line), and the 2-layer model presented in this paper employing the $\Delta_{Linear}=0.009$ coefficients (dashed line).



Figure 18: Vertical profile of horizontal velocity (top row) and vertical velocity (bottom row) under the crest of a sine wave for three different kh values, as given by linear theory (solid line), the high-order model of Gobbi *et al.* (2000) (dotted line), and the 2-layer model presented in this paper employing the $\Delta_{Linear}=0.02$ coefficients (dashed line).

Table 2: β values from nonlinear optimization for two-layer model.

Ω (kh)	β_1	β_2	eta_3	$\Delta_{Nonlinear}$
5	0.176	0.113	-0.013	0.013
10	0.124	-0.044	-0.068	0.036

the minimization parameter $\Delta_{Nonlinear}$:

$$\Delta_{Nonlinear} = \frac{1}{2} \left(\frac{\sum_{kh=1}^{\Omega} \frac{|\zeta_{Stokes}^{(1)} - \zeta^{(1)}|}{kh}}{\sum_{kh=1}^{\Omega} \frac{|\zeta_{Stokes}^{(1)}|}{kh}} + \frac{\sum_{k_1h=1}^{\Omega} \sum_{k_2h=1}^{\Omega} \frac{|\zeta_{Stokes}^{\mp} - \zeta^{\mp}|}{k_1h + k_2h}}{\sum_{k_1h=1}^{\Omega} \sum_{k_2h=1}^{\Omega} \frac{|\zeta_{Stokes}^{\mp} - \zeta^{\mp}|}{k_1h + k_2h}} \right)$$
(48)

Note that the summation limit for the nonlinear error begins at kh=1.0. The extremely large values of these nonlinear parameters at kh values less than one lead to poor error quantifications at higher wavenumbers when using this type of error formulation. A summary of the nonlinear optimization results is shown in Table 2.

Figure 19 shows the second-order free surface correction associated with these two sets of β , along with the correction with no nonlinear optimization, i.e. $\beta_1 = \beta_2 = \beta_3 = 0$. The $\Delta_{Nonlinear}=0.013$ shows excellent agreement to kh of 6, where the relative error is just over 5%. After this point, the error grows continuously. For the $\Delta_{Nonlinear}=0.036$ optimization, 5% errors are found at a kh near 3, although the error is less for high kh as compared to the $\Delta_{Nonlinear}=0.013$ case. The bichromatic transfer amplitudes are shown in Figure 20, where the superharmonics are given in the upper left, and the subharmonics the lower right. For the case with no nonlinear optimization, Fig. 20a), good agreement is only found at small kh values for both super- and subharmonics. However, with some nonlinear optimization, as shown in Fig. 20a) for $\Delta_{Nonlinear}=0.013$, the superharmonic amplitudes become much more accurate. In fact, transfers where k_1 and k_2 are close show excellent agreement, with the 5% error contour extending to $k_1 = k_2 = 5.5$. However, subharmonic transfer are relatively unaffected by the optimization, and lose accurate quickly for k_1 values greater than 3. With respect to the subharmonic amplitudes, the same can be said for the $\Delta_{Nonlinear}=0.036$ optimization as well, shown in Fig. 20c). Accurate superharmonics are predicted slightly better for this optimization, where the 5% error contour extends to $k_1 = k_2 = 6$. It is noted that these transfer plots show very similar behavior to those given by Kennedy et al. (2001) for the nonlinear-optimized, one-layer model. In fact, the error of the two-layer model, for the $\Delta_{Nonlinear}=0.013$ case, is approximately 1/2 of the one-layer model error at all (k_1, k_2) combinations. As with the linear optimization, choosing which set of β values are best to use depends on the specifics of the problem to be examined. The authors choose to employ the set of coefficients from the $\Delta_{Nonlinear}=0.013$ case. This set exhibits significantly better accuracy at all wavenumbers less than 5, which is a highly desirable characteristic.

Nonlinear optimization is performed to second-order only. To optimize the model to thirdorder, for example, best results would be achieved by continuation of the nonlinear expansion



Figure 19: Second-order free surface correction, $\zeta^{(1)}$, relative to the Stokes solution, with no nonlinear optimization ($\beta_1 = \beta_2 = \beta_3=0$) shown by the dash-dotted line, the $\Delta_{Nonlinear}=0.013$ results by the dotted line, and the $\Delta_{Nonlinear}=0.036$ results by the dashed line.



Figure 20: Sub- and superharmonic transfer amplitudes for bichromatic wave interactions, ζ^{\mp} , relative to the Stokes solution, where subharmonics are shown in the lower right, and superharmonics in the upper left. Shown in a) are the results with no nonlinear optimization $(\beta_1 = \beta_2 = \beta_3 = 0)$, in b) the $\Delta_{Nonlinear} = 0.013$ results, and the $\Delta_{Nonlinear} = 0.036$ results in c).

of the evaluation levels, i.e. the third-order expansion for the layer boundary would take the form

$$\eta = \alpha_2 h + \beta_2 \zeta + \frac{(\gamma_2 \zeta)^2}{h} \tag{49}$$

The coefficients γ would then be tuned such that an optimal agreement with third-order Stokes theory is obtained.

13.3 Three-Layer Equation Model

For the three-layer model, the horizontal velocity vectors are given in nondimensional form as

$$\boldsymbol{U}_{3} = \boldsymbol{u}_{3} - \mu_{3}^{2} \left\{ \frac{z_{3}^{2} - \kappa_{3}^{2}}{2} \nabla S_{3} + (z_{3} - \kappa_{3}) \nabla T_{3} \right\} + O(\mu_{3}^{4})$$
(50)

$$\boldsymbol{U}_{2} = \boldsymbol{u}_{2} - \mu_{2}^{2} \left\{ \frac{z_{2}^{2} - \kappa_{2}^{2}}{2} \nabla S_{2} + (z_{2} - \kappa_{2}) \nabla T_{2} \right\} + O(\mu_{2}^{4}, \mu_{2}^{2} \mu_{3}^{2})$$
(51)

$$\boldsymbol{U}_{1} = \boldsymbol{u}_{1} - \mu_{1}^{2} \left\{ \frac{z_{1}^{2} - \kappa_{1}^{2}}{2} \nabla S_{1} + (z_{1} - \kappa_{1}) \nabla T_{1} \right\} + O(\mu_{1}^{4}, \mu_{1}^{2} \mu_{2}^{2}, \mu_{1}^{2} \mu_{3}^{2})$$
(52)

where

$$S_{3} = \frac{d_{3}}{h_{o}} \nabla \cdot \boldsymbol{u}_{3}, \qquad T_{3} = \nabla \cdot (h\boldsymbol{u}_{3}) + \frac{1}{\varepsilon_{o}} \frac{\partial h}{\partial t}$$

$$S_{2} = \frac{d_{2}}{h_{o}} \nabla \cdot \boldsymbol{u}_{2}, \qquad T_{2} = \eta_{2} \left(\frac{b_{2}}{d_{3}}S_{3} - \frac{b_{2}}{d_{2}}S_{2}\right) + T_{3}$$

$$S_{1} = \frac{d_{1}}{h_{o}} \nabla \cdot \boldsymbol{u}_{1}, \qquad T_{1} = \eta_{1} \left(\frac{d_{1}}{d_{2}}S_{2} - S_{1}\right) + T_{2} \qquad (53)$$

Table 3: α values from linear optimization for three-layer model.

			-		*	
$\Omega ~({ m kh})$	α_1	α_2	$lpha_3$	$lpha_4$	$lpha_5$	Δ_{Linear}
10	-0.083	-0.156	-0.315	-0.494	-0.748	0.0003
20	-0.079	-0.149	-0.310	-0.492	-0.749	0.0005
[6,6] Pade :	-0.13145	-0.24292	-0.42545	-0.62015	-0.83693	

The evaluation levels are defined as:

$$\kappa_1 = \alpha_1 h, \quad \eta_1 = \alpha_2 h, \quad \kappa_2 = \alpha_3 h, \quad \eta_2 = \alpha_4 h, \quad \kappa_3 = \alpha_5 h$$
(54)

The continuity, momentum, and matching equations for the three-layer system are as given in the previous chapter.

13.3.1 Choice of Arbitrary Levels: Linear Optimization

For the three- and more layer systems, only the linear dispersion properties will be examined in this thesis. The dispersion relation for the three-layer model takes the form:

$$w^{2} = \frac{k^{2}gh\left[1 + (kh)^{2}N_{1}^{(3)} + (kh)^{4}N_{2}^{(3)} + (kh)^{6}N_{3}^{(3)}\right]}{1 + (kh)^{2}D_{1}^{(3)} + (kh)^{4}D_{2}^{(3)} + (kh)^{6}D_{3}^{(3)}}$$
(55)

The coefficients $N^{(3)}$ and $D^{(3)}$ are tedious functions of the α values. These coefficients were calculated using the symbolic math package *Macsyma*, and are given in Appendix A.

For this analysis, the minimization error, Δ_{Linear} , is now given by:

$$\Delta_{Linear} = \frac{1}{2} \left(\frac{\sum_{kh=0.1}^{\Omega} \frac{|c^e - c|}{kh}}{\sum_{kh=0.1}^{\Omega} \frac{|c^e|}{kh}} + \frac{\sum_{kh=0.1}^{\Omega} \frac{|c^e_g - c_g|}{kh}}{\sum_{kh=0.1}^{\Omega} \frac{|c^e_g|}{kh}} \right)$$
(56)

where the shoaling error is no longer taken into account. A summary of the optimization results is shown in Table 3. Only two Ω values (or Δ_{Linear} values) are looked at, due to the computational requirements of the optimization. The significant CPU time arises due simply to the fact that the minimization is performed on a five-dimensional function, where each of the five free parameters is determined to three significant digits. The phase and group velocity of the three-layer model is shown in Figure 21. The three-layer model has very good accuracy to $kh \approx 15$, which is a significant improvement over the two-layer model. In order for the three-layer model to be applied to practical engineering problems, shoaling and nonlinear properties need to be examined. This examination is not done in this thesis, however, the analysis is feasible, yet extraordinarily complex and tedious.



Figure 21: Comparison of wave speed and group velocity of the three-layer model with the exact linear relation; the dashed-dotted line is the [6,6] Pade, the dotted line is the [8,8] Pade, the dashed line is the three-layer results with $\Delta_{Linear}=0.0003$, and the slide line is the three-layer results with $\Delta_{Linear}=0.0005$.

13.3.2Vertical Velocity Profiles

Let us define the function $f_1(z)$ as the horizontal velocity, with constant water depth, normalized by its value at z = 0. This function is composed of three quadratic polynomial elements, given by:

$$f_{1}(z) = \frac{1 + (kh)^{2} \left\{ \frac{1}{2} \left(\frac{z^{2}}{h^{2}} - \alpha_{1}^{2} \right) + \left(\alpha_{1} - \frac{z}{h} \right) \left[\alpha_{2} - \frac{u_{2}^{(0)}}{u_{1}^{(0)}} (\alpha_{2} - \alpha_{4}) - \frac{u_{3}^{(0)}}{u_{1}^{(0)}} (\alpha_{4} + 1) \right] \right\}}{1 + (kh)^{2} \left\{ -\frac{1}{2} \alpha_{1}^{2} + \alpha_{1} \left[\alpha_{2} - \frac{u_{2}^{(0)}}{u_{1}^{(0)}} (\alpha_{2} - \alpha_{4}) - \frac{u_{3}^{(0)}}{u_{1}^{(0)}} (\alpha_{4} + 1) \right] \right\}},$$
for $z \ge \eta_{1} = \alpha_{2}h$
(57)

$$f_{1}(z) = f_{1}(\eta_{1}) \frac{1 + (kh)^{2} \left\{ \frac{1}{2} \left(\frac{z^{2}}{h^{2}} - \alpha_{3}^{2} \right) + \left(\alpha_{3} - \frac{z}{h} \right) \left[\alpha_{4} - \frac{u_{3}^{(0)}}{u_{2}^{(0)}} (\alpha_{4} + 1) \right] \right\}}{1 + (kh)^{2} \left\{ \frac{1}{2} \left(\alpha_{2}^{2} - \alpha_{3}^{2} \right) + \left(\alpha_{3} - \alpha_{2} \right) \left[\alpha_{4} - \frac{u_{3}^{(0)}}{u_{2}^{(0)}} (\alpha_{4} + 1) \right] \right\}},$$

$$for \ \eta_{2} \leq z < \eta_{1}$$

$$(58)$$

$$f_1(z) = f_1(\eta_2) \frac{1 + (kh)^2 \left\{ \frac{1}{2} \left(\frac{z}{h^2} - \alpha_5^2 \right) + \left(\frac{z}{h} - \alpha_5 \right) \right\}}{1 + (kh)^2 \left\{ \frac{1}{2} \left(\alpha_4^2 - \alpha_5^2 \right) + \left(\alpha_4 - \alpha_5 \right) \right\}},$$

for $z < \eta_2 = \alpha_4 h$ (59)

From the linear equation system we have explicit expressions for $u_n^{(0)}$, and thus the ratios $u_2^{(0)}/u_1^{(0)}$, $u_3^{(0)}/u_1^{(0)}$, and $u_3^{(0)}/u_2^{(0)}$ can be evaluated. Similarly, the vertical velocity profile, normalized by the velocity at the still water level,

is given by $f_2(z)$:

$$f_{2}(z) = \frac{z/h - \alpha_{2} + u_{2}^{(0)}/u_{1}^{(0)}(\alpha_{2} - \alpha_{4}) + u_{3}^{(0)}/u_{1}^{(0)}(\alpha_{4} + 1)}{-\alpha_{2} + u_{2}^{(0)}/u_{1}^{(0)}(\alpha_{2} - \alpha_{4}) + u_{3}^{(0)}/u_{1}^{(0)}(\alpha_{4} + 1)},$$

for $z \ge \eta = \alpha_{2}h$ (60)

$$f_2(z) = f_2(\eta_1) \frac{z/h - \alpha_4 + u_3^{(0)}/u_2^{(0)}(\alpha_4 + 1)}{\alpha_2 - \alpha_4 + u_3^{(0)}/u_2^{(0)}(\alpha_4 + 1)}, \quad \text{for } \eta_2 \le z < \eta_1$$
(61)

$$f_2(z) = f_2(\eta_2) \frac{z/h+1}{\alpha_4+1}, \quad \text{for } z < \eta_2 = \alpha_4 h$$
 (62)

which is a piecewise linear function.

Figures 22 and 22 compare the three-layer vertical profile of velocity to linear theory. The overall agreement is very good to near kh=15, where errors in the vertical velocity profile become large. The horizontal velocity profile is well captured to $kh \approx 15$.



Figure 22: Vertical profiles of velocity for three-layer model (dashed line), compared with linear theory (solid line). The top row shows horizontal velocity and the bottom vertical velocity. The three-layer profiles use the α values from the $\Delta_{Linear}=0.0003$ optimization.



Figure 23: Figure setup same as in Fig 22, except here showing high kh comparisons.

13.4 Four-Layer Equation Model

For the four-layer model, the horizontal velocity vectors are given as

$$\boldsymbol{U}_{4} = \boldsymbol{u}_{4} - \mu_{4}^{2} \left\{ \frac{z_{4}^{2} - \kappa_{4}^{2}}{2} \nabla S_{4} + (z_{4} - \kappa_{4}) \nabla T_{4} \right\} + O(\mu_{4}^{4})$$
(63)

$$\boldsymbol{U}_{3} = \boldsymbol{u}_{3} - \mu_{3}^{2} \left\{ \frac{z_{3}^{2} - \kappa_{3}^{2}}{2} \nabla S_{3} + (z_{3} - \kappa_{3}) \nabla T_{3} \right\} + O(\mu_{3}^{4}, \mu_{3}^{2} \mu_{4}^{2})$$
(64)

$$\boldsymbol{U}_{2} = \boldsymbol{u}_{2} - \mu_{2}^{2} \left\{ \frac{z_{2}^{2} - \kappa_{2}^{2}}{2} \nabla S_{2} + (z_{2} - \kappa_{2}) \nabla T_{2} \right\} + O(\mu_{2}^{4}, \mu_{2}^{2} \mu_{3}^{2}, \mu_{2}^{2} \mu_{4}^{2})$$
(65)

$$\boldsymbol{U}_{1} = \boldsymbol{u}_{1} - \mu_{1}^{2} \left\{ \frac{z_{1}^{2} - \kappa_{1}^{2}}{2} \nabla S_{1} + (z_{1} - \kappa_{1}) \nabla T_{1} \right\} + O(\mu_{1}^{4}, \mu_{1}^{2} \mu_{2}^{2}, \mu_{1}^{2} \mu_{3}^{2}, \mu_{1}^{2} \mu_{4}^{2})$$
(66)

where

$$S_{4} = \frac{d_{4}}{h_{o}} \nabla \cdot \boldsymbol{u}_{4}, \qquad T_{4} = \nabla \cdot (h\boldsymbol{u}_{4}) + \frac{1}{\varepsilon_{o}} \frac{\partial h}{\partial t}$$

$$S_{3} = \frac{d_{3}}{h_{o}} \nabla \cdot \boldsymbol{u}_{3}, \qquad T_{3} = \eta_{3} \left(\frac{b_{3}}{d_{4}}S_{4} - \frac{b_{3}}{d_{3}}S_{3}\right) + T_{4}$$

$$S_{2} = \frac{d_{2}}{h_{o}} \nabla \cdot \boldsymbol{u}_{2}, \qquad T_{2} = \eta_{2} \left(\frac{b_{2}}{d_{3}}S_{3} - \frac{b_{2}}{d_{2}}S_{2}\right) + T_{3}$$

$$S_{1} = \frac{d_{1}}{h_{o}} \nabla \cdot \boldsymbol{u}_{1}, \qquad T_{1} = \eta_{1} \left(\frac{d_{1}}{d_{2}}S_{2} - S_{1}\right) + T_{2} \qquad (67)$$

The evaluation levels are defined as:

$$\kappa_1 = \alpha_1 h, \quad \eta_1 = \alpha_2 h, \quad \kappa_2 = \alpha_3 h, \quad \eta_2 = \alpha_4 h,$$

$$\kappa_3 = \alpha_5 h, \quad \eta_3 = \alpha_6 h, \quad \kappa_4 = \alpha_7 h \tag{68}$$

The continuity, momentum, and matching equations for the four-layer system are as given in the previous chapter.

13.4.1 Choice of Arbitrary Levels: Linear Optimization

The dispersion relation for the four-layer model takes the form:

$$w^{2} = \frac{k^{2}gh\left[1 + (kh)^{2}N_{1}^{(4)} + (kh)^{4}N_{2}^{(4)} + (kh)^{6}N_{3}^{(4)} + (kh)^{8}N_{4}^{(4)}\right]}{1 + (kh)^{2}D_{1}^{(4)} + (kh)^{4}D_{2}^{(4)} + (kh)^{6}D_{3}^{(4)} + (kh)^{8}D_{4}^{(4)}}$$
(69)

The coefficients $N^{(4)}$ and $D^{(4)}$ are tedious functions of the α values. These coefficients were calculated using the symbolic math package Macsyma, and are given in Appendix B.

Table 4: α values from linear optimization for four-layer model.

$\Omega ~({ m kh})$	α_1	α_2	$lpha_3$	
20	-0.0575	-0.1080	-0.2086	
α_4	α_5	$lpha_6$	α_7	Δ_{Linear}

As with the analysis of the three-layer model, the minimization error, Δ_{Linear} , is given by:

$$\Delta_{Linear} = \frac{1}{2} \left(\frac{\sum_{kh=0.1}^{\Omega} \frac{|c^e - c|}{kh}}{\sum_{kh=0.1}^{\Omega} \frac{|c^e|}{kh}} + \frac{\sum_{kh=0.1}^{\Omega} \frac{|c^e_g - c_g|}{kh}}{\sum_{kh=0.1}^{\Omega} \frac{|c^e_g|}{kh}} \right)$$
(70)

where the shoaling error is no longer taken into account. A summary of the optimization results is shown in Table 4. Only one Ω value (or Δ_{Linear} value) is looked at, due to the extreme computational requirements of the optimization. The significant CPU time arises due to the fact that the minimization is performed on a seven-dimensional function, where each of the seven free parameters is determined to four significant digits. Additionally, expression of the four-layer dispersion relation is quite tedious. For example, coding only the expression for $N_4^{(4)}$ in (69), which is a function of $\alpha_1, ..., \alpha_7$, requires 170 lines of FORTRAN code. The α coefficients are determined to four significant digits for the four-layer equation model, whereas in all the previous analysis only three digits are found, because the dispersion relation is sensitive to these digits. This sensitivity is due to the high powers of kh in (69), which require more precise coefficients to optimize the dispersion relation.

The phase and group velocity of the three-layer model is shown in Figure 24. The fourlayer model has very good accuracy to $kh \approx 25$. Examination and optimization of shoaling, and in particular nonlinear, properties of the four-layer model is nearly an insurmountable task with current computational abilities, due to the number of free parameters and the complexity of the functions to be optimized.

13.4.2 Vertical Velocity Profiles

Let us define the function $f_1(z)$ as the horizontal velocity, with constant water depth, normalized by its value at z = 0. This function is composed of four quadratic polynomial elements, given by:

$$f_1(z) = \left(1 + (kh)^2 \left\{ \frac{1}{2} \left(\frac{z^2}{h^2} - \alpha_1^2\right) + \left(\alpha_1 - \frac{z}{h}\right) \left[\alpha_2 - \frac{u_2^{(0)}}{u_1^{(0)}}(\alpha_2 - \alpha_4) - \frac{u_3^{(0)}}{u_1^{(0)}}(\alpha_4 - \alpha_6) - \frac{u_4^{(0)}}{u_1^{(0)}}(\alpha_6 + 1)\right] \right\} \right) / \\ 1 + (kh)^2 \left\{ -\frac{1}{2}\alpha_1^2 + \alpha_1 \left[\alpha_2 - \frac{u_2^{(0)}}{u_1^{(0)}}(\alpha_2 - \alpha_4) - \frac{u_3^{(0)}}{u_1^{(0)}}(\alpha_4 - \alpha_6) - \frac{u_4^{(0)}}{u_1^{(0)}}(\alpha_6 + 1)\right] \right\},$$



Figure 24: Comparison of wave speed and group velocity of the four-layer model with the exact linear relation; the dashed-dotted line is the [6,6] Pade, the dotted line is the [8,8] Pade, the dashed line is the four-layer results with $\Delta_{Linear}=0.0003$.

$$for \ z \ge \eta_1 = \alpha_2 h \tag{71}$$

$$f_1(z) =$$

$$f_1(\eta_1) \frac{1 + (kh)^2 \left\{ \frac{1}{2} \left(\frac{z^2}{h^2} - \alpha_3^2 \right) + \left(\alpha_3 - \frac{z}{h} \right) \left[\alpha_4 - \frac{u_3^{(0)}}{u_2^{(0)}} (\alpha_4 - \alpha_6) - \frac{u_4^{(0)}}{u_2^{(0)}} (\alpha_6 + 1) \right] \right\}, \qquad (72)$$

$$f_1(\eta_1) \frac{1 + (kh)^2 \left\{ \frac{1}{2} (\alpha_2^2 - \alpha_3^2) + (\alpha_3 - \alpha_2) \left[\alpha_4 - \frac{u_3^{(0)}}{u_2^{(0)}} (\alpha_4 - \alpha_5) - \frac{u_4^{(0)}}{u_2^{(0)}} (\alpha_6 + 1) \right] \right\}, \qquad (72)$$

$$f_1(z) = f_1(\eta_2) \frac{1 + (kh)^2 \left\{ \frac{1}{2} \left(\frac{z^2}{h^2} - \alpha_5^2 \right) + \left(\alpha_5 - \frac{z}{h} \right) \left[\alpha_6 - \frac{u_4^{(0)}}{u_3^{(0)}} (\alpha_6 + 1) \right] \right\}, \qquad (73)$$

$$f_1(z) = f_1(\eta_3) \frac{1 + (kh)^2 \left\{ \frac{1}{2} (\alpha_4^2 - \alpha_5^2) + (\alpha_5 - \alpha_4) \left[\alpha_6 - \frac{u_4^{(0)}}{u_3^{(0)}} (\alpha_6 + 1) \right] \right\}, \qquad (73)$$

$$f_1(z) = f_1(\eta_3) \frac{1 + (kh)^2 \left\{ \frac{1}{2} \left(\frac{z^2}{h^2} - \alpha_7^2 \right) + \left(\frac{z}{h} - \alpha_7 \right) \right\}}{1 + (kh)^2 \left\{ \frac{1}{2} (\alpha_6^2 - \alpha_7^2) + (\alpha_6 - \alpha_7) \right\}, \qquad (74)$$

From the linear equation system we have explicit expressions for $u_n^{(0)}$, and thus the various velocity ratios can be can be evaluated.

Similarly, the vertical velocity profile, normalized by the velocity at the still water level, is given by $f_2(z)$:

$$f_{2}(z) = \frac{z/h - \alpha_{2} + u_{2}^{(0)}/u_{1}^{(0)}(\alpha_{2} - \alpha_{4}) + u_{3}^{(0)}/u_{1}^{(0)}(\alpha_{4} - \alpha_{6}) + u_{4}^{(0)}/u_{1}^{(0)}(\alpha_{6} + 1)}{-\alpha_{2} + u_{2}^{(0)}/u_{1}^{(0)}(\alpha_{2} - \alpha_{4}) + u_{3}^{(0)}/u_{1}^{(0)}(\alpha_{4} - \alpha_{6}) + u_{4}^{(0)}/u_{1}^{(0)}(\alpha_{6} + 1)},$$

$$for \ z \ge \eta = \alpha_{2}h$$

$$z/h - \alpha_{4} + u_{2}^{(0)}/u_{2}^{(0)}(\alpha_{4} - \alpha_{6}) + u_{4}^{(0)}/u_{2}^{(0)}(\alpha_{6} + 1)$$
(75)

$$f_{2}(z) = f_{2}(\eta_{1}) \frac{z/h - \alpha_{4} + u_{3}^{(0)}/u_{2}^{(0)}(\alpha_{4} - \alpha_{6}) + u_{4}^{(0)}/u_{2}^{(0)}(\alpha_{6} + 1)}{\alpha_{2} - \alpha_{4} + u_{3}^{(0)}/u_{2}^{(0)}(\alpha_{4} - \alpha_{6}) + u_{4}^{(0)}/u_{2}^{(0)}(\alpha_{6} + 1)},$$

for $\alpha_{4}h = \eta_{2} \le z < \eta_{1} = \alpha_{2}h$ (76)

$$f_{2}(z) = f_{2}(\eta_{2}) \frac{z/h - \alpha_{6} + u_{4}^{(0)}/u_{3}^{(0)}(\alpha_{6} + 1)}{\alpha_{4} - \alpha_{6} + u_{4}^{(0)}/u_{3}^{(0)}(\alpha_{6} + 1)},$$

for $\alpha_{6}h = \eta_{3} \le z < \eta_{2} = \alpha_{4}h$ (77)

$$f_2(z) = f_2(\eta_3) \frac{z/h+1}{\alpha_6+1}, \quad \text{for } z < \eta_3 = \alpha_6 h$$
 (78)

which is a piecewise linear function.

Figures 25 and 26 compare the three-layer vertical profile of velocity to linear theory. The overall agreement is very good to near kh=25, where errors in the vertical velocity profile become large.



Figure 25: Vertical profiles of velocity for four-layer model (dashed line), compared with linear theory (solid line). The top row shows horizontal velocity and the bottom vertical velocity.



Figure 26: Figure setup same as in Fig 25, except here showing high kh comparisons.

13.5 Summary

Through linear and nonlinear optimization of the interface and velocity evaluation locations, it is shown that the two-layer model exhibits accurate linear characteristics up to a $kh \approx$ 8 and nonlinear accuracy to $kh \approx 6$. This is a greater than two-fold extension to higher kh over existing $O(\mu_o^2)$ Boussinesq-type models, while maintaining the maximum order of differentiation at three. A less thorough optimization of the three- and four-layer models is undertaken, examining only phase and group velocity. This optimization indicates that the three-layer model equations are accurate to $kh \approx 15$ and the four layer-model to $kh \approx 25$. Figure 27 summarizes the results from this chapter. This figure gives the phase and group velocity for the two-, three-, and four-layer models, as well as the traditional and high-order Boussinesq models. The most striking feature of this plot is the disproportionate increase in accurate from the two-layer model to the three-layer model. This feature certainly requires a more in-depth investigation of the three-layer model in the near future.



Figure 27: Comparison of wave speed and group velocity for numerous different models. Curve (1) is the [2,2] Pade properties used by some Boussinesq models, (2) is the [4,4] Pade of the high-order Boussinesq model, (3) is the two-layer model, (4) is the three-layer model, and (5) is the four-layer model.

14 Appendix B. Numerical Model for One- & Two-Layer Systems

14.1 Numerical Scheme

In this section, a finite difference algorithm is presented for the general one- and two-layer model equations. The structure of the present numerical model is similar to those of Wei & Kirby (1995) and Wei *et al.* (1995). Differences between the model presented here, for the one-layer system, and that of Wei *et al.* exist in the added terms due to a time-dependant water depth and the numerical treatment of some nonlinear dispersive terms, which will be discussed in more detail. A high-order predictor-corrector scheme is utilized, employing a third order in time explicit Adams-Bashforth predictor step, and a fourth order in time Adams-Moulton implicit corrector step (Press *et al.*, 1989). The implicit corrector step must be iterated until a convergence criterion is satisfied. The governing equations are dimensionalized for the numerical model, and all variables described in this and following sections will be in the dimensional form. Note that the dimensional equations are equivalent to the non-dimensional ones with $\varepsilon = \mu = 1$ and the addition of gravity, g, to the coefficient of the leading order free surface derivative in the momentum equation.

14.2 Numerical Expressions for One-Layer System

To simplify the predictor-corrector equations, the velocity time derivatives in the momentum equations are grouped into the dimensional form:

$$\mathcal{U} = u + \frac{\kappa^2 - \zeta^2}{2} u_{xx} + (\kappa - \zeta)(hu)_{xx} - \zeta_x \left[\zeta u_x + (hu)_x\right]$$
(79)

$$\mathcal{V} = v + \frac{\kappa^2 - \zeta^2}{2} v_{yy} + (\kappa - \zeta) (hv)_{yy} - \zeta_y \left[\zeta v_y + (hv)_y \right]$$
(80)

where subscripts denote partial derivatives. Note that this grouping is different from that given in Wei *et al.* (1995). The grouping given above in (79) and (80) incorporates nonlinear terms, which is not done in Wei *et al.*. These nonlinear time derivatives arise from the nonlinear dispersion terms $\nabla \left[\zeta (\nabla \cdot (h \boldsymbol{u}_{\alpha})_t + \frac{h_{tt}}{\varepsilon}) \right]$ and $\nabla \left(\frac{\zeta^2}{2} \nabla \cdot \boldsymbol{u}_{\alpha t} \right)$, which can be reformulated using the relation:

$$\nabla \left[\zeta (\nabla \cdot (h\boldsymbol{u}_{\alpha})_{t} + \frac{h_{tt}}{\varepsilon}) \right] = \nabla \left[\zeta (\nabla \cdot (h\boldsymbol{u}_{\alpha}) + \frac{h_{t}}{\varepsilon}) \right]_{t} - \nabla \left[\zeta_{t} (\nabla \cdot (h\boldsymbol{u}_{\alpha}) + \frac{h_{t}}{\varepsilon}) \right]$$
$$\nabla \left(\frac{\zeta^{2}}{2} \nabla \cdot \boldsymbol{u}_{\alpha t} \right) = \nabla \left(\frac{\zeta^{2}}{2} \nabla \cdot \boldsymbol{u}_{\alpha} \right)_{t} - \nabla \left(\zeta \zeta_{t} \nabla \cdot \boldsymbol{u}_{\alpha} \right)$$
(81)

The author has found that this form is more stable and requires less iterations to converge for highly nonlinear problems, as compared to the Wei *et al.* formulation. The predictor equations are

$$\zeta_{i,j}^{n+1} = \zeta_{i,j}^n + \frac{\Delta t}{12} (23E_{i,j}^n - 16E_{i,j}^{n-1} + 5E_{i,j}^{n-2})$$
(82)

$$\mathcal{U}_{i,j}^{n+1} = \mathcal{U}_{i,j}^n + \frac{\Delta t}{12} (23F_{i,j}^n - 16F_{i,j}^{n-1} + 5F_{i,j}^{n-2}) + 2(F_1)_{i,j}^n - 3(F_1)_{i,j}^{n-1} + (F_1)_{i,j}^{n-2}$$
(83)

$$\mathcal{V}_{i,j}^{n+1} = \mathcal{V}_{i,j}^n + \frac{\Delta t}{12} (23G_{i,j}^n - 16G_{i,j}^{n-1} + 5G_{i,j}^{n-2}) + 2(G_1)_{i,j}^n - 3(G_1)_{i,j}^{n-1} + (G_1)_{i,j}^{n-2}$$
(84)

where

$$E = -h_t - [(\zeta + h)u]_x - [(\zeta + h)v]_y + \left\{ (h + \zeta) \left[\left(\frac{1}{6} \left(\zeta^2 - \zeta h + h^2 \right) - \frac{1}{2} \kappa^2 \right) S_x + \left(\frac{1}{2} \left(\zeta - h \right) - \kappa \right) T_x \right] \right\}_x + \left\{ (h + \zeta) \left[\left(\frac{1}{6} \left(\zeta^2 - \zeta h + h^2 \right) - \frac{1}{2} \kappa^2 \right) S_y + \left(\frac{1}{2} \left(\zeta - h \right) - \kappa \right) T_y \right] \right\}_y$$

$$F = -\frac{1}{2} [(u^2)_x + (v^2)_x] - q\zeta_x - \kappa h_{xtt} - \kappa t h_{xt}$$
(85)

$$+ (Eh_{t} + \zeta h_{tt})_{x} - [E(\zeta S + T)]_{x} - \left[\frac{1}{2} \left(\kappa^{2} - \zeta^{2}\right) \left(uS_{x} + vS_{y}\right)\right]_{x} - [(\kappa - \zeta) \left(uT_{x} + vT_{y}\right)]_{x} - \frac{1}{2} \left[(T + \zeta S)^{2}\right]_{x}$$
(86)

$$F_1 = \frac{\zeta^2 - \kappa^2}{2} v_{xy} - (\kappa - \zeta)(hv)_{xy} + \zeta_x [\zeta v_y + (hv)_y]$$
(87)

$$G = -\frac{1}{2}[(u^{2})_{y} + (v^{2})_{y}] - g\zeta_{y} - \kappa h_{ytt} - \kappa_{t}h_{yt}$$
$$+ (Eh_{t} + \zeta h_{tt})_{y} - [E(\zeta S + T)]_{y} - \left[\frac{1}{2}\left(\kappa^{2} - \zeta^{2}\right)\left(uS_{x} + vS_{y}\right)\right]_{y}$$
$$- [(\kappa - \zeta)\left(uT_{x} + vT_{y}\right)]_{y} - \frac{1}{2}\left[\left(T + \zeta S\right)^{2}\right]_{y}$$
(88)

$$G_1 = \frac{\zeta^2 - \kappa^2}{2} u_{xy} - (\kappa - \zeta)(hu)_{xy} + \zeta_y[\zeta u_x + (hu)_x]$$
(89)

and

$$S = u_x + v_y$$
 $T = (hu)_x + (hv)_y + h_t$ (90)

All first order spatial derivatives are differenced with fourth order ($\Delta x^4 = \Delta y^4$) accurate equations, which are five-point differences. Second order spatial derivatives are approximated with three-point centered finite difference equations, which are second order accurate. The second order spatial derivatives are taken to lower order accuracy because these derivatives only appear in dispersive terms. The "combined" dispersive-numerical error for the second order derivatives is $O(\Delta x^2 \mu_o^2)$, which is less than the error associated with dispersive truncation error of the equations, $O(\mu_o^4)$, as long as $\Delta x < h$, which will generally be the case. This will not be the case as h approaches zero, however, as this occurs the problem becomes a shallow water problem, and μ_o approaches zero as well. Terms are evaluated at the local grid point (i, j), and n represents the current time step, when values of ζ , u and v are known. The above expressions, (85) - (90), are for the fully nonlinear problem; if a weakly nonlinear or non-dispersive system is to be examined, the equations should be truncated accordingly. The fourth-order implicit corrector expressions for the free surface elevation and horizontal velocities are

$$\zeta_{i,j}^{n+1} = \zeta_{i,j}^n + \frac{\Delta t}{24} (9E_{i,j}^{n+1} + 19E_{i,j}^n - 5E_{i,j}^{n-1} + E_{i,j}^{n-2})$$
(91)

$$\mathcal{U}_{i,j}^{n+1} = \mathcal{U}_{i,j}^n + \frac{\Delta t}{24} (9F_{i,j}^{n+1} + 19F_{i,j}^n - 5F_{i,j}^{n-1} + F_{i,j}^{n-2}) + (F_1)_{i,j}^{n+1} - (F_1)_{i,j}^n$$
(92)

$$\mathcal{V}_{i,j}^{n+1} = \mathcal{V}_{i,j}^n + \frac{\Delta t}{24} (9G_{i,j}^{n+1} + 19G_{i,j}^n - 5G_{i,j}^{n-1} + G_{i,j}^{n-2}) + (G_1)_{i,j}^{n+1} - (G_1)_{i,j}^n \tag{93}$$

The system is solved by first evaluating the predictor equations, then u and v are solved via (79) and (80), respectively. Both (79) and (80) yield a diagonal matrix after finite differencing. The matrices are diagonal, with a bandwidth of three (due to three-point finite differencing), and the efficient Thomas algorithm can be utilized. At this point in the numerical system, we have predictors for ζ , u, and v. Next, the corrector expressions are evaluated, and again u and v are determined from (79) and (80). The error is calculated, in order to determine if the implicit correctors need to be reiterated. The error criteria employed is a dual calculation, and requires that either

$$max \left| \frac{w^{n+1} - w_*^{n+1}}{w^{n+1}} \right| < \frac{\epsilon}{100} \quad or \quad \frac{\sum |w^{n+1} - w_*^{n+1}|}{\sum |w^{n+1}|} < \epsilon \tag{94}$$

be satisfied for the iteration to stop. In the above, w represents ζ , u, and v, and w_* is the previous iterations value. The expression on the left represents a maximum local error, while the right is the average local error over the entire domain. The error threshold, ϵ , is set to 10^{-6} . For the local error calculation, it is noted that inevitably there will be locations in the numerical domain where values of the physical variables are close to zero, and applying this error calculation to these points may lead to unnecessary iterations in the corrector loop. Thus it is required that $|\frac{\zeta}{a}|, |\frac{u,v}{\epsilon\sqrt{gh}}| > 10^{-4}$ for the corresponding error calculation to proceed. Linear stability analysis for this numerical model as been performed by Wei (1995), as well as Hsiao (2000) and Woo (2002), and will not be repeated here. This analysis tells that $\Delta t < \frac{\Delta x}{2c}$ to ensure stability, where c is the wave celerity. For the numerical exterior boundaries, two types of conditions are applied: reflective and

For the numerical exterior boundaries, two types of conditions are applied: reflective and radiation. The reflective, or no-flux, boundary condition for the Boussinesq equations has been examined by previous researchers (e.g., Wei & Kirby, 1995), and their methodology is followed here. For the radiation, or open, boundary condition, a sponge layer is utilized. The sponge layer is applied in the manner recommended by Kirby *et al.* (1998).

14.3 Numerical Expressions for Two-Layer System

The velocity time derivatives in the momentum equation are grouped into the dimensional form:

$$\mathcal{U} = u_1 + \left[\frac{\kappa_1^2 - 2\kappa_1 \eta - \zeta^2 + 2\zeta \eta}{2} \frac{\partial^2 u_1}{\partial x^2} + \left(\eta \frac{\partial \zeta}{\partial x} + \zeta \frac{\partial \eta}{\partial x} - \kappa_1 \frac{\partial \eta}{\partial x} - \zeta \frac{\partial \zeta}{\partial x} \right) \frac{\partial u_1}{\partial x} \right] = 0$$
(95)

$$\mathcal{V} = v_1 + \left[\frac{\kappa_1^2 - 2\kappa_1\eta - \zeta^2 + 2\zeta\eta}{2}\frac{\partial^2 v_1}{\partial y^2} + \left(\eta\frac{\partial\zeta}{\partial y} + \zeta\frac{\partial\eta}{\partial y} - \kappa_1\frac{\partial\eta}{\partial y} - \zeta\frac{\partial\zeta}{\partial y}\right)\frac{\partial v_1}{\partial y}\right] = 0$$
(96)

where subscripts denote partial derivatives. The predictor equations are identical to the one-layer equations:

$$\zeta_{i,j}^{n+1} = \zeta_{i,j}^n + \frac{\Delta t}{12} (23E_{i,j}^n - 16E_{i,j}^{n-1} + 5E_{i,j}^{n-2})$$
(97)

$$\mathcal{U}_{i,j}^{n+1} = \mathcal{U}_{i,j}^n + \frac{\Delta t}{12} (23F_{i,j}^n - 16F_{i,j}^{n-1} + 5F_{i,j}^{n-2}) + 2(F_1)_{i,j}^n - 3(F_1)_{i,j}^{n-1} + (F_1)_{i,j}^{n-2}$$
(98)

$$\mathcal{V}_{i,j}^{n+1} = \mathcal{V}_{i,j}^n + \frac{\Delta t}{12} (23G_{i,j}^n - 16G_{i,j}^{n-1} + 5G_{i,j}^{n-2}) + 2(G_1)_{i,j}^n - 3(G_1)_{i,j}^{n-1} + (G_1)_{i,j}^{n-2} \tag{99}$$

where

$$E = -h_t - \left[(\zeta - \eta)u_1 + (\eta + h)u_2 \right]_x - \left[(\zeta - \eta)v_1 + (\eta + h)v_2 \right]_y + \left\{ \left[\frac{\zeta^3 - \eta^3}{6} - \frac{(\zeta - \eta)\kappa_1^2}{2} \right] S_{1x} + \left[\frac{\zeta^2 - \eta^2}{2} - (\zeta - \eta)\kappa_1 \right] T_{1x} \right\}_x + \left\{ \left[\frac{\eta^3 + h^3}{6} - \frac{(\eta + h)\kappa_2^2}{2} \right] S_{2x} + \left[\frac{\eta^2 - h^2}{2} - (\eta + h)\kappa_2 \right] T_{2x} \right\}_x + \left\{ \left[\frac{\zeta^3 - \eta^3}{6} - \frac{(\zeta - \eta)\kappa_1^2}{2} \right] S_{1y} + \left[\frac{\zeta^2 - \eta^2}{2} - (\zeta - \eta)\kappa_1 \right] T_{1y} \right\}_y + \left\{ \left[\frac{\eta^3 + h^3}{6} - \frac{(\eta + h)\kappa_2^2}{2} \right] S_{2y} + \left[\frac{\eta^2 - h^2}{2} - (\eta + h)\kappa_2 \right] T_{2y} \right\}_y$$
(100)

$$F = -\frac{1}{2} [(u_1^2)_x + (v_1^2)_x] - g\zeta_x - [E(\zeta S_1 + T_1)]_x - \left[\frac{1}{2} \left(\kappa_1^2 - \zeta^2\right) (u_1 S_{1_x} + v_1 S_{1_y})\right]_x - \left[(\kappa_1 - \zeta) \left(u_1 T_{1_x} + v_1 T_{1_y}\right)\right]_x - \frac{1}{2} \left[(T_1 + \zeta S_1)^2\right]_x$$
(101)

$$F_{1} = -(\kappa_{1} - \zeta) \left[\eta \left(S_{2} - v_{1_{y}} \right) + T_{2} \right]_{x} - \frac{\kappa_{1}^{2} - \zeta^{2}}{2} v_{1_{xy}} + \zeta \zeta_{x} v_{1_{y}} + \zeta_{x} \left[\eta \left(S_{2} - v_{1_{y}} \right) + T_{2} \right]$$
(102)

$$G = -\frac{1}{2} [(u_1^2)_y + (v_1^2)_y] - g\zeta_y - [E(\zeta S_1 + T_1)]_y - \left[\frac{1}{2} \left(\kappa_1^2 - \zeta^2\right) (u_1 S_{1_x} + v_1 S_{1_y})\right]_y - \left[(\kappa_1 - \zeta) \left(u_1 T_{1_x} + v_1 T_{1_y}\right)\right]_y - \frac{1}{2} \left[(T_1 + \zeta S_1)^2\right]_y$$
(103)

$$G_{1} = -(\kappa_{1} - \zeta) \left[\eta \left(S_{2} - u_{1x} \right) + T_{2} \right]_{y} - \frac{\kappa_{1}^{2} - \zeta^{2}}{2} u_{1xy} + \zeta \zeta_{y} u_{1x} + \zeta_{y} \left[\eta \left(S_{2} - u_{1x} \right) + T_{2} \right]$$
(104)

and

$$S_{1} = u_{1_{x}} + v_{1_{y}} \qquad T_{1} = \eta(S_{2} - S_{1}) + T_{2}$$

$$S_{2} = u_{2_{x}} + v_{2_{y}} \qquad T_{2} = (hu_{2})_{x} + (hv_{2})_{y} + h_{t} \qquad (105)$$

The fourth-order implicit corrector expressions for the free surface elevation and horizontal velocities are

$$\zeta_{i,j}^{n+1} = \zeta_{i,j}^n + \frac{\Delta t}{24} (9E_{i,j}^{n+1} + 19E_{i,j}^n - 5E_{i,j}^{n-1} + E_{i,j}^{n-2})$$
(106)

$$\mathcal{U}_{i,j}^{n+1} = \mathcal{U}_{i,j}^{n} + \frac{\Delta t}{24} (9F_{i,j}^{n+1} + 19F_{i,j}^{n} - 5F_{i,j}^{n-1} + F_{i,j}^{n-2}) + (F_1)_{i,j}^{n+1} - (F_1)_{i,j}^{n}$$
(107)

$$\mathcal{V}_{i,j}^{n+1} = \mathcal{V}_{i,j}^n + \frac{\Delta t}{24} (9G_{i,j}^{n+1} + 19G_{i,j}^n - 5G_{i,j}^{n-1} + G_{i,j}^{n-2}) + (G_1)_{i,j}^{n+1} - (G_1)_{i,j}^n \tag{108}$$

The lower layer velocities are determined from the equation:

$$u_{2} + \left\{ \frac{\kappa_{2}^{2} + \eta^{2} - 2\eta\kappa_{1}}{2} u_{2xx} + (\kappa_{2} - \kappa_{1}) (hu_{2})_{xx} + (\eta - \kappa_{1}) \eta_{x} u_{2x} \right\} = u_{1} + \left\{ \frac{(\kappa_{1} - \eta)^{2}}{2} S_{1x} + (\kappa_{1} - \eta) \left[h_{xt} + \eta_{x} \left(v_{2y} - S_{1} \right) \right] + \frac{2\eta\kappa_{1} - \kappa_{2}^{2} - \eta^{2}}{2} v_{2xy} + (\kappa_{1} - \kappa_{2}) (hv_{2})_{xy} \right\}$$
(109)
$$v_{2} + \left\{ \frac{\kappa_{2}^{2} + \eta^{2} - 2\eta\kappa_{1}}{2} v_{2yy} + (\kappa_{2} - \kappa_{1}) (hv_{2})_{yy} + (\eta - \kappa_{1}) \eta_{y} v_{2y} \right\} = v_{1} + \left\{ \frac{(\kappa_{1} - \eta)^{2}}{2} S_{1y} + (\kappa_{1} - \eta) \left[h_{yt} + \eta_{y} \left(u_{2x} - S_{1} \right) \right] + \frac{2\eta\kappa_{1} - \kappa_{2}^{2} - \eta^{2}}{2} u_{2xy} + (\kappa_{1} - \kappa_{2}) (hu_{2})_{xy} \right\}$$
(110)

Energy Dissipation Mechanisms 14.4

2

Two forms of physical dissipation are considered in the numerical model for one-layer only, wave breaking and bottom friction. These mechanisms modify the momentum equation:

$$\frac{\partial \boldsymbol{u}_1}{\partial t} + \dots + \boldsymbol{R}_f - \boldsymbol{R}_b = 0 \tag{111}$$

where R_f accounts for bottom friction dissipation and R_b for wave breaking. The evaluation of these two additional terms will be discussed in this section. The dissipation terms have only to date been utilized by the one-layer model, although two-layer dissipation will be examined in future work.

Table 5: Relation between C and f for various roughnesses.

$C(s/m^{1/2})$	f
10	0.1
15	0.044
30	0.011
60	0.0027

14.4.1 Bottom Friction

Bottom friction is described in the quadratic form:

$$\boldsymbol{R}_f = \frac{f}{H} \boldsymbol{u}_b |\boldsymbol{u}_b| \tag{112}$$

where f is a bottom friction coefficient, typically in the range of 10^{-3} to 10^{-2} (e.g. Whitfor and Thorton, 1996; Kobayashi et al., 1997), depending on the Reynolds number and seafloor condition, $H = h + \zeta$, the total water depth, and u_b is the horizontal velocity at the seafloor. The above expression, (112), has been utilized in similar models (e.g., Chen at al, 1999) and has a direct correlation to the Chezy coefficient, C. This relationship is:

$$f = \frac{g}{C^2} \tag{113}$$

where g is gravity. Table 5 shows a few conversions between C and f. The low C value of 10 can be thought of as the "rough-beach limit" (Mader, 1990), and C values of 20-60 are typical for river channels. Unless otherwise noted, the simulations presented in this thesis use a bottom friction value of 0.005.

14.4.2 Wave Breaking Model

14.4.3 Previous Work

One of the most significant obstacles in the way of developing a practical numerical model with depth-integrated equations is wave breaking. A depth-integrated model, by definition, can only have a single elevation value of the water-air interface at any horizontal coordinate, and thus phenomena such as wave overturning cannot be simulated. Along the same lines, very strong horizontal vorticity typically accompanies breaking, which an irrotational or weakly rotational model will not capture. Most depth-integrated derivations use as an initial assumption inviscid flow (those in this thesis included), and therefore do not have any means to dissipate energy. These three reasons constitute the major problems with wave breaking in depth-integrated models, although the first given, that of the impossibility of simulating wave-overturning, is the only unapproachable one of the three. Thus, it will always be necessary to parameterize the large-scale features of wave breaking when using depth-integrated equations.

Two distinct approaches to simulating the effects of wave breaking with depth-integrated models exist: numerical dissipation and ad-hoc addition of dissipative terms to the momentum equation. Numerical dissipative approaches most notably include shock capturing schemes. In these schemes, energy dissipation is related to the local smoothness of the solution, which is of course strongly related to the grid length near the shock. Most recently, Li and Raichlen (2002) used the weighted essentially non-oscillatory shock capturing scheme, adapted from gas dynamics research, to model solitary wave runup. The results presented in Li and Raichlen are excellent, among the best numerical-experimental comparisons to date. With shock capturing methods, the numerical results tend to be very smooth. However, the dissipation is entirely numerical, and although the general form of the dissipative terms may be of the proper physical form, the dissipation will inevitable be related to the grid length and time step.

Utilizing post-derivation-added dissipation terms to the momentum equation removes this dissipative dependence on numerical parameters. However, these added terms are ad-hoc terms, and will contain coefficients that must be obtained based on comparison with experiment. Whether the numerical or ad-hoc approach is more desirable will depend entirely on the individual preference of the researcher. In this thesis, the addition of ad-hoc dissipative and dispersive enhacements/errors whenever possible.

14.4.4 Breaking Scheme and Validation

The breaking scheme employed in this thesis work closely follows the scheme presented in Kennedy *et al.* (2000). Description of this particular breaking scheme can also be found in Chen *et al.* (2000), which is a companion paper to Kennedy *et al.*. The scheme is developed from an "eddy viscosity" approach, where a user-defined formulation for an eddy viscosity is developed based soley on agreement with experimental data. The eddy viscosity is part of a momentum conserving, ad-hoc dissipative term, $\mathbf{R}_b = R_{bx} \mathbf{i} + R_{by} \mathbf{j}$, where:

$$R_{bx} = \frac{1}{H} \left\{ [\nu(Hu_1)_x]_x + \frac{1}{2} [\nu(Hu_1)_y + \nu(Hv_1)_x]_y \right\},$$
(114)

$$R_{by} = \frac{1}{H} \left\{ [\nu(Hv_1)_y]_y + \frac{1}{2} [\nu(Hv_1)_x + \nu(Hu_1)_y]_x \right\},$$
(115)

 ν is the eddy viscosity, and $H = h + \zeta$, the total water depth. The above expressions are identical to those found in Kennedy *et al.*. Eddy viscosity is calculated as:

$$\nu = BH\zeta_t \tag{116}$$

The purpose of the variable B is to ensure a smooth transition between breaking and nonbreaking states. The formulation developed and employed by Kennedy et al. is:

$$B = \begin{cases} \delta, & \zeta_t \ge 2\zeta_t^b\\ \delta\left(\zeta_t/\zeta_t^b - 1\right), & \zeta_t^b < \zeta_t \le 2\zeta_t^b\\ 0, & \zeta_t \le \zeta_t^b \end{cases}$$

where δ is some amplification factor and the parameter ζ_t^b determines the onset and stoppage of breaking. ζ_t^b is evaluated as

$$\zeta_t^b = \begin{cases} \zeta_t^{(F)}, & t - t_o \ge T^b \\ \zeta_t^{(I)} + \frac{t - t_o}{T^b} \left(\zeta_t^{(F)} - \zeta_t^{(I)}\right), & 0 \le t - t_o < T^b \end{cases}$$

Trial	Period (s)	Height (cm)
031041	3.33	4.3
041041	2.5	3.9
051041	2.0	3.6
061071	1.67	6.7
A10112	1.0	6.7

Table 6: Experimental wave characteristics for the Hansen and Svendsen tests.

where $\zeta_t^{(I)}$ is the initial free surface transient threshold that must be exceeded for a breaking event to initiate, $\zeta_t^{(F)}$ is the minimum transient required for a breaking event to continue, t is the local time, t_o is the time breaking started, and T^b is a transition time. There is no physical evidence to support this formulation for the eddy viscosity, ν , and it is chosen entirely on its ability to recreate experimental results to an accurate degree. Up to this point, the breaking model is identical to that of Kennedy *et al.*. The difference lies in the evaluation of the free parameters, of which there are four. In Kennedy *et al.*, the parameters are based on the linear long wave speed, i.e. $\zeta_t^{(I)} = 0.65\sqrt{gh}$. Determination of the parameters in this fashion is undesirable for the model presented in this thesis, because this model calculates the free surface as it runs up a shoreline, where *h* is negative. Thus, evaluation of the parameters in areas where h < 0 would require an additional specification. The simplest method to eliminate this problem is to utilize the nonlinear long wave speed = \sqrt{gH} . Using the nonlinear long wave speed also requires repeating all of the wave breaking analysis in Kennedy *et al.*, to determine to optimum value of the four free parameters.

Hansen and Svendsen (1979) performed a number of regular wave tests on plane slopes. Five of these experiments are recreated numerically, described in Table 6. The waves were generated in 0.36 *m* of water, and shoaled up a 1:34.26 slope. Time series were taken at numerous locations along the wave flume; wave height and mean free surface elevation will be compared here. Through trial and error minimization of the difference between numerical and experimental results, the following set of free parameters is chosen: $\delta = 6.5$, $\zeta_t^{(I)} = 0.65\sqrt{gH}$, $\zeta_t^{(F)} = 0.08\sqrt{gH}$, and $T^b = 8.0\sqrt{H/g}$. Figures 28 - 32 show the numerical - experimental comparisons for the five cases. For all cases, the agreement is very good, with all exhibiting significant improvement over the corresponding comparisons in Kennedy *et al.*. There is a clear pattern in the numerics to predict the initiation of breaking slightly earlier (in deeper water) than occurred in the experiments. This pattern is also evident in the Kennedy *et al.* results, where it is postulated to be caused by the known overprediction of nonlinear superharmonics by the one-layer model.

There are a few other minor differences between the breaking model implementation presented in this thesis and that of Kennedy *et al.*. In Kennedy *et al.*, the eddy viscosity as calculated by (116) is spatially filtered using a three-point filter before it is inserted into (115). Filtering the eddy viscosity was found to have no advantageous effect when using the modified formulation presented in this thesis, and was not performed. Additionally, use of a spatial filter, filtering the calculated free surface and velocity values, was found to be unnecessary, and no filtering has been performed in any of the simulations presented in this paper.



Figure 28: Experimental (dots) and numerical (line) wave height and mean free surface for Hansen and Svendsen case 031041.



Figure 29: Experimental (dots) and numerical (line) wave height and mean free surface for Hansen and Svendsen case 041041.



Figure 30: Experimental (dots) and numerical (line) wave height and mean free surface for Hansen and Svendsen case 051041.



Figure 31: Experimental (dots) and numerical (line) wave height and mean free surface for Hansen and Svendsen case 061071.



Figure 32: Experimental (dots) and numerical (line) wave height and mean free surface for Hansen and Svendsen case 10112.
When examining the comparisons in Figures 28 - 32, one needs to keep in mind the numerical treatment of the shoreline. Accurate modeling of the wave reflection off the beach is an integral part of accurate prediction of the wave height, particularly near the break point. In the next section, the moving boundary scheme used by the numerical model is described.

15 Appendix C. Derivation and Numerical Model for Weakly Rotational, Finite Volume Method

15.1 Dimensionless Governing Equations

The basic approach for including viscous effects into the Boussinesq equations is to derive the governing equations not from Eulers equations but from the Navier-Stokes equations. For the derivation of the approximate, depth-integrated model, a non-dimensionalization, or scaling, of the primitive equations is the first step. Consistent with previous Boussinesq-type approaches, it is expected that the leading order solution will be shallow water, and thus a long wave scaling is used. A spatial region is characterized by a typical water depth h_o , a horizontal length scale ℓ_o , wave amplitude a_o , and a time scale $\ell_o/\sqrt{gh_o}$. With the variables, the following dimensionless variables and a parameter can be introduced.

$$(x,y) = \frac{(x',y')}{\ell_o}, \qquad z = \frac{z'}{h_o}, \qquad t = \frac{t'\sqrt{gh_o}}{\ell_o}, \qquad h = \frac{h'}{h_o}, \qquad \zeta = \frac{\zeta'}{h_o}, \\ (U,V) = \frac{(U',V')}{\sqrt{gh_o}}, \qquad W = \frac{W'}{\mu\sqrt{gh_o}}, \qquad p = \frac{p'}{\rho gh_o}, \qquad \mu = \frac{h_o}{\ell_o}$$
(117)

where (x', y') denotes horizontal axes, z' is a vertical axis, t' is time, h' is water depth, ζ' is water surface elevation, (U', V') are horizontal velocities, W' is a vertical direction velocity, and p' is a pressure. The g and ρ are a gravitational acceleration and density, respectively. All these variables are dimensional. The μ is a standard parameter for a scale analysis of long waves.

For this study, due to the depth-integration and resulting loss of flow details in the vertical plane, it will be reasonable to divide the turbulent eddy viscosity into horizontal and vertical components, as is commonly done for shallow mixing studies. The Smagorinsky model (1963) will be used for the horizontal eddy viscosity $\nu_t^{h'}$, that is, $\nu_t^{h'} = (C_s \Delta')^2 \sqrt{2S'_{ij}S'_{ij}}$ where C_s is a constant, the S'_{ij} is a strain rate tensor and Δ' is the grid size. By applying the above scalings to the horizontal eddy viscosity, it can be expressed as

$$\nu_t^{h'} = C_s^2 \Delta^2 h_o \sqrt{gh_o} \sqrt{\left(\frac{\partial u}{\partial z}\right)^2 + 2\mu^2 \left(\frac{\partial u}{\partial x}\right)^2 + 2\mu^2 \left(\frac{\partial w}{\partial z}\right)^2 + \dots}$$
(118)

Equation (2) is rewritten in the compact form

$$\nu_t^{h'} = \alpha h_o \sqrt{g h_o} \nu_t^h \tag{119}$$

where $\alpha = C_s^2 \Delta^2$. For the vertical eddy viscosity, we presume a shallow flow formulation, where the vertical turbulence is driven by the bottom shear only. The expression $\nu_t^{v'} = C_h H' u'_*$ is used where the H' is the total water depth and the u'_* is the friction velocity. The vertical eddy viscosity can be non-dimensionalized as

$$\nu_t^{\nu\prime} = \beta h_o \sqrt{gh_o} H u_b = \beta h_o \sqrt{gh_o} \nu_t^{\nu} \tag{120}$$

where u_b represents a near-bottom free stream velocity such that $u_* = C_* u_b$ and β is equal to $C_h C_*$.

Finally, the continuity equation and the Navier-Stokes equations can be scaled with the equations (1), (3) and (4):

$$\nabla \cdot \boldsymbol{U} + \frac{\partial W}{\partial z} = 0 \tag{121}$$

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} + W \frac{\partial \boldsymbol{U}}{\partial z} + \nabla p = \alpha \mu \nabla \cdot \left(\nu_t^h \nabla \boldsymbol{U}\right) + \frac{\beta}{\mu} \frac{\partial}{\partial z} \left(\nu_t^v \frac{\partial \boldsymbol{U}}{\partial z}\right)$$
(122)

$$\mu^{2} \frac{\partial W}{\partial t} + \mu^{2} \boldsymbol{U} \cdot \nabla W + \mu^{2} W \frac{\partial W}{\partial z} + \frac{\partial p}{\partial z} + 1 = \alpha \mu^{3} \nabla \cdot \left(\nu_{t}^{h} \nabla W\right) + \beta \mu \frac{\partial}{\partial z} \left(\nu_{t}^{v} \frac{\partial W}{\partial z}\right)$$
(123)

15.2 Derivation of the Depth Integrated Momentum Equations

This derivation will be of perturbation type, and a small parameter assumption must be made. Looking to the vertical momentum equation (7), it is assumed that $O(\mu^2) = O(\mu\beta) \ll$ 1, yielding

$$\frac{\partial p}{\partial z} + 1 = O(\mu^2, \mu\beta) \tag{124}$$

The above indicates that to leading order, the pressure is hydrostatic, which will permit the standard depth integration to obtain a long wave model. Thus the derived model will be restricted to weakly dispersive waves and flow with weak vertical turbulence.

Typically, the perturbation of the inviscid primitive equations is performed using μ^2 as the small parameter. In these inviscid cases, where of course $\alpha = \beta = 0$, the small parameter choice essentially required by (8) is clear. Here, with viscosity, the choice is not clear, as either μ^2 or $\mu\beta$ could be used as the small parameter. Mathematically, there is no reason to choose one over the other, as in fact both would result in the same final equations. For the derivation presentation, μ^2 will be used, and this issue of ambiguity will be addressed later.

Physical values are expanded with power series following

$$f = \sum_{k=1}^{N} \mu^{(2n)} f_n \tag{125}$$

where f = p, U, V, W and μ^2 assumed to be small. Substituting this expansion into (7) or (8) gives p_o as hydrostatic. It follows that ∇p_o is independent of z. This implies that in the horizontal momentum equation, all the other leading order should also be z-independent functions (Dellar and Salmon, 2005). Consequently, \boldsymbol{U}_o becomes $\boldsymbol{U}_o(x, y, t)$.

At the water surface and at the bottom, the following boundary conditions $W_{\zeta} = \partial \zeta / \partial t + U_{\zeta} \cdot \nabla \zeta$ at $z = \zeta$ and $W_{-h} + U_{-h} \cdot \nabla h = 0$ at z = -h can be applied. The vertical velocity can be expressed with the horizontal velocity terms by integrating the continuity equation, yielding,

$$W_o = -zS - T \tag{126}$$

where $S = \nabla \cdot \boldsymbol{U}_o$ and $T = \nabla \cdot (h\boldsymbol{U}_o)$.

With the perturbation analysis, the horizontal vorticity is expressed as

$$\frac{\partial U'}{\partial z'} - \nabla W' = \mu^2 \frac{c_o}{h_o} \left(\frac{\partial U_1}{\partial z} - \nabla W_o \right) + \frac{c_o}{h_o} O\left(\mu^4\right) = \mu^2 \frac{c_o}{h_o} \boldsymbol{\omega}_1 + \frac{c_o}{h_o} O\left(\mu^4\right)$$
(127)

A vertical profile of U_1 can be derived from equation (11) through a vertical integration:

$$\boldsymbol{U}_{1} = -\frac{1}{2}z^{2}\nabla S - z\nabla T + \frac{1}{2}h^{2}\nabla S - h\nabla T + \int_{-h}^{z} \boldsymbol{\omega}_{1}dz + \boldsymbol{U}_{1}(-h) + O\left(\mu^{2}\right)$$
(128)

such that the horizontal velocity, including up to second-order terms, becomes

$$\boldsymbol{U} = \boldsymbol{U}_{o} - \mu^{2} \left(\frac{1}{2} z^{2} \nabla S + z \nabla T - \frac{1}{2} h^{2} \nabla S + h \nabla T \right) + \mu^{2} \int_{-h}^{z} \boldsymbol{\omega}_{1} dz + \mu^{2} \boldsymbol{U}_{1}(-h) + O\left(\mu^{4}\right)$$
(129)

As this derivation will make use of Nwogu's (1993) approach, the horizontal velocity is evaluated at an arbitrary elevation $z = z_{\alpha}$,

$$\boldsymbol{U}_{\alpha} = \boldsymbol{U}_{o} - \mu^{2} \left(\frac{1}{2} z_{\alpha}^{2} \nabla S + z_{\alpha} \nabla T - \frac{1}{2} h^{2} \nabla S + h \nabla T \right) + \mu^{2} \int_{-h}^{z_{\alpha}} \boldsymbol{\omega}_{1} dz + \mu^{2} \boldsymbol{U}_{1}(-h) + O\left(\mu^{4}\right)$$
(130)

Subtracting the equation (14) from the equation (13), U can be expressed in terms of U_{α} .

$$\boldsymbol{U} = \boldsymbol{U}_{\alpha} + \mu^{2} \left\{ \frac{1}{2} \left(z_{\alpha}^{2} - z^{2} \right) \nabla S + (z_{\alpha} - z) \nabla T \right\} + \mu^{2} \boldsymbol{\Omega} + O\left(\mu^{4} \right)$$
(131)

where $\mathbf{\Omega} = \int_{z_{\alpha}}^{z} \boldsymbol{\omega}_{1} dz$. For later use, the horizontal velocity can be expressed as $\boldsymbol{U} = \boldsymbol{U}_{\alpha} + \mu^{2} \left(\boldsymbol{U}_{1}^{\phi} + \boldsymbol{U}_{1}^{r} \right) + O\left(\mu^{4}\right)$ in which $\boldsymbol{U}_{1}^{r} = \boldsymbol{\Omega}$ and $\boldsymbol{U}_{1}^{\phi} = \left(U_{1}^{\phi}, V_{1}^{\phi} \right)$ is defined as

$$\boldsymbol{U}_{1}^{\phi} = \frac{1}{2} \left(z_{\alpha}^{2} - z^{2} \right) \nabla S + (z_{\alpha} - z) \nabla T$$
(132)

The vertical profile of pressure is found through integration of the vertical momentum equation. Noting that the vertical distribution of ν_t^v is independent on z as shown in the equation (4), the pressure can be expressed as

$$p = \zeta - z$$

$$+ \mu^{2} \frac{1}{2} \left(z^{2} - \zeta^{2} \right) \frac{\partial S}{\partial t} + \mu^{2} \left(z - \zeta \right) \frac{\partial T}{\partial t}$$

$$+ \mu^{2} \frac{1}{2} \left(z^{2} - \zeta^{2} \right) \boldsymbol{U}_{o} \cdot \nabla S + \mu^{2} \left(z - \zeta \right) \boldsymbol{U}_{o} \cdot \nabla T$$

$$+ \mu^{2} \frac{1}{2} \left(\zeta^{2} - z^{2} \right) S^{2} + \mu^{2} \left(\zeta - z \right) TS$$

$$+ O \left(\mu^{4}, \alpha \mu^{3}, \beta \mu^{3} \right)$$
(133)

The next step in deriving the horizontal depth-integrated momentum equation is to express each term of the horizontal momentum equations through U_{α} . These terms, included to elucidate how vorticity and viscosity terms appear, become

$$\frac{\partial U}{\partial t} = \frac{\partial U_{\alpha}}{\partial t} + \mu^2 \frac{\partial}{\partial t} \left\{ \frac{1}{2} \left(z_{\alpha}^2 - z^2 \right) \nabla S + (z_{\alpha} - z) \nabla T \right\} + \mu^2 \frac{\partial \Omega}{\partial t} + O\left(\mu^4\right)$$
(134)

$$\boldsymbol{U} \cdot \nabla \boldsymbol{U} = \boldsymbol{U}_{\alpha} \cdot \nabla \boldsymbol{U}_{\alpha} + \mu^{2} \nabla \left[\boldsymbol{U}_{\alpha} \cdot \left\{ \frac{1}{2} \left(z_{\alpha}^{2} - z^{2} \right) \nabla S + (z_{\alpha} - z) \nabla T \right\} \right] + \mu^{2} \nabla \left(\boldsymbol{U}_{\alpha} \cdot \boldsymbol{\Omega} \right) + \mu^{2} \boldsymbol{\xi} + O\left(\mu^{4} \right)$$
(135)

$$W\frac{\partial \boldsymbol{U}}{\partial z} = \mu^2 \left(z^2 S \nabla S + z T \nabla S + z S \nabla T + T \nabla T + W_o \boldsymbol{\omega}_1 \right) + O\left(\mu^4\right)$$
(136)

$$\nabla p = \nabla \zeta$$

$$- \mu^{2} \frac{1}{2} \nabla \left(\zeta^{2} \frac{\partial S}{\partial t} \right) - \mu^{2} \nabla \left(\zeta \frac{\partial T}{\partial t} \right) + \mu^{2} \frac{1}{2} \nabla \left(z^{2} \frac{\partial S}{\partial t} \right) + \mu^{2} \nabla \left(z \frac{\partial T}{\partial t} \right)$$

$$- \mu^{2} \frac{1}{2} \nabla \left(\zeta^{2} \boldsymbol{U}_{\alpha} \cdot \nabla S \right) - \mu^{2} \nabla \left(\zeta \boldsymbol{U}_{\alpha} \cdot \nabla T \right) + \mu^{2} \nabla \left(\frac{1}{2} \zeta^{2} S^{2} \right) + \mu^{2} \nabla \left(\zeta T S \right)$$

$$+ \mu^{2} \frac{1}{2} \nabla \left(z^{2} \boldsymbol{U}_{\alpha} \cdot \nabla S \right) + \mu^{2} \nabla \left(z \boldsymbol{U}_{\alpha} \cdot \nabla T \right) - \mu^{2} \nabla \left(\frac{1}{2} z^{2} S^{2} \right) - \mu^{2} \nabla \left(z T S \right)$$

$$+ O \left(\mu^{4} \right)$$
(137)

$$\alpha\mu\nabla\cdot\left(\nu_{t}^{h}\nabla\boldsymbol{U}\right) = \alpha\mu\nabla\cdot\left(\nu_{t}^{h}\nabla\boldsymbol{U}_{\alpha}\right) + O\left(\alpha\mu^{3}\right)$$
(138)

$$\frac{\beta}{\mu}\frac{\partial}{\partial z}\left(\nu_t^v\frac{\partial \boldsymbol{U}}{\partial z}\right) = \beta\mu\frac{\partial\nu_t^v\boldsymbol{\omega}_1}{\partial z} - \beta\mu\nu_t^v\nabla S + O\left(\beta\mu^3\right)$$
(139)

In the equation (19), $\boldsymbol{\xi} = (\xi^x, \xi^y)$ is defined as

$$\xi^{x} = \left[V_{\alpha} \left\{ \frac{\partial \left(U_{1}^{\phi} + \Omega^{x} \right)}{\partial y} - \frac{\partial \left(V_{1}^{\phi} + \Omega^{y} \right)}{\partial x} \right\} - \left(V_{1}^{\phi} + \Omega^{y} \right) \left(\frac{\partial V_{\alpha}}{\partial x} - \frac{\partial U_{\alpha}}{\partial y} \right) \right]$$
(140)

$$\xi^{y} = \left[U_{\alpha} \left\{ \frac{\partial \left(V_{1}^{\phi} + \Omega^{y} \right)}{\partial x} - \frac{\partial \left(U_{1}^{\phi} + \Omega^{x} \right)}{\partial y} \right\} + \left(U_{1}^{\phi} + \Omega^{x} \right) \left(\frac{\partial V_{\alpha}}{\partial x} - \frac{\partial U_{\alpha}}{\partial y} \right) \right]$$
(141)

where $(U_{\alpha}, V_{\alpha}) = U_{\alpha}$ and Ω^x and Ω^y are defined as $\Omega = (\Omega^x, \Omega^y)$.

The horizontal vorticity term appearing in equation (23), namely $\partial \nu_t^v \omega_1 / \partial z$, can be expressed through a shear stress, $\boldsymbol{\tau}$, in the following way:

$$\frac{\partial \nu_t^v \boldsymbol{\omega}_1}{\partial z} = \frac{\partial}{\partial z} \left\{ \nu_t^v \left(\frac{\partial \boldsymbol{U}_1^r}{\partial z} + \frac{\partial \boldsymbol{U}_1^\phi}{\partial z} - \nabla W_o \right) \right\} = \frac{\partial}{\partial z} \left(\nu_t^v \frac{\partial \boldsymbol{U}_1^r}{\partial z} \right) = \frac{\partial \boldsymbol{\tau}}{\partial z}$$
(142)

If the shear stress is assumed to vary linearly from zero at the water surface to τ_b at the bottom (Rodi, 1980), then the horizontal vorticity terms can be expressed as

$$\boldsymbol{\omega}_1 = \frac{\partial \boldsymbol{U}_1^r}{\partial z} = \frac{\boldsymbol{\tau}_b}{\nu_t^v} \frac{\zeta - z}{\zeta + h} \tag{143}$$

$$\mathbf{\Omega} = \int_{z_{\alpha}}^{z} \boldsymbol{\omega}_{1} dz = \frac{\boldsymbol{\tau}_{b}}{\nu_{t}^{v} \left(\zeta + h\right)} \left\{ \frac{1}{2} \left(z_{\alpha}^{2} - z^{2} \right) + \zeta \left(z - z_{\alpha} \right) \right\}$$
(144)

Equation (28) shows that with horizontal vorticity correlated directly, and simply, to the bottom stress, the depth-integrated result

By substituting the derived equations (18)-(28) into the equation (6), the depth-integrated momentum equation becomes:

$$\frac{\partial U_{\alpha}}{\partial t} + U_{\alpha} \cdot \nabla U_{\alpha} + \nabla \zeta$$

$$- \mu^{2} \frac{1}{2} \nabla \left(\zeta^{2} \frac{\partial S}{\partial t} \right) - \mu^{2} \nabla \left(\zeta \frac{\partial T}{\partial t} \right) + \mu^{2} \left(\frac{1}{2} z_{\alpha}^{2} \frac{\partial \nabla S}{\partial t} + z_{\alpha} \frac{\partial \nabla T}{\partial t} \right)$$

$$- \mu^{2} \frac{1}{2} \nabla \left(\zeta^{2} U_{\alpha} \cdot \nabla S \right) - \mu^{2} \nabla \left(\zeta U_{\alpha} \cdot \nabla T \right) + \mu^{2} \nabla \left(\frac{1}{2} \zeta^{2} S^{2} \right) + \mu^{2} \nabla \left(\zeta T S \right)$$

$$+ \mu^{2} \frac{1}{2} \nabla \left(z_{\alpha}^{2} U_{\alpha} \cdot \nabla S \right) + \mu^{2} \nabla \left(z_{\alpha} U_{\alpha} \cdot \nabla T \right) + \mu^{2} \left(T \nabla T \right)$$

$$+ \mu^{2} \frac{\partial}{\partial t} \left[\psi \left\{ \frac{1}{2} \left(z_{\alpha}^{2} - z^{2} \right) + \zeta \left(z - z_{\alpha} \right) \right\} \right]$$

$$+ \mu^{2} \left(U_{\alpha} \cdot \left[\psi \left\{ \frac{1}{2} \left(z_{\alpha}^{2} - z^{2} \right) + \zeta \left(z - z_{\alpha} \right) \right\} \right] \right)$$

$$- \mu^{2} \left(zS + T \right) \left(\zeta - z \right) \psi + \mu^{2} \xi$$

$$- \alpha \mu \nabla \cdot \left(\nu_{t}^{h} \nabla U_{\alpha} \right) + \beta \mu \nu_{t}^{\nu} \nabla S - \beta \mu \frac{\partial \tau}{\partial z}$$

$$= O \left(\mu^{4}, \alpha \mu^{3}, \beta \mu^{3} \right)$$
(145)

where $\boldsymbol{\psi} = \boldsymbol{\tau}_b / \{ \nu_t^v \left(\zeta + h \right) \}.$

15.3 Elimination of z-dependent Terms

Several approaches have been used to eliminate the z-dependent terms in the Boussinesq type momentum equations. In Hsiao et al.(2002) and in many publications, irrotational flow assumptions were used to eliminate the terms. Chen et al. (2003) eliminated the z dependency by setting $z = z_{\alpha}$ in these terms. In this study, the approach proposed by Chen (2006) is used; the equation (31) is depth-averaged. For example, the τ term can be rewritten by

$$\frac{1}{\zeta+h} \int_{-h}^{\zeta} \frac{\partial \boldsymbol{\tau}}{\partial z} dz = -\frac{\boldsymbol{\tau}_b}{\zeta+h} \tag{146}$$

Finally, the depth-integrated momentum equations including viscosity and vorticity effects can be expressed as

$$\frac{\partial \boldsymbol{U}_{\alpha}}{\partial t} + \boldsymbol{U}_{\alpha} \cdot \nabla \boldsymbol{U}_{\alpha} + \nabla \zeta + \mu^{2} \left(\boldsymbol{D} + \boldsymbol{D}^{\nu} + \overline{\boldsymbol{\xi}} + \overline{\boldsymbol{\xi}^{\nu}} \right)
- \alpha \mu \nabla \cdot \left(\nu_{t}^{h} \nabla \boldsymbol{U}_{\alpha} \right) + \beta \mu \nu_{t}^{v} \nabla S + \beta \mu \frac{\boldsymbol{\tau}_{b}}{\zeta + h}
= O \left(\mu^{4}, \alpha \mu^{3}, \beta \mu^{3} \right)$$
(147)

where

$$D = \frac{1}{2} \nabla \left(z_{\alpha}^{2} \boldsymbol{U}_{\alpha} \cdot \nabla S \right) + \nabla \left(z_{\alpha} \boldsymbol{U}_{\alpha} \cdot \nabla T \right) + (T \nabla T)$$

$$- \frac{1}{2} \nabla \left(\zeta^{2} \frac{\partial S}{\partial t} \right) - \nabla \left(\zeta \frac{\partial T}{\partial t} \right) + \left(\frac{1}{2} z_{\alpha}^{2} \frac{\partial \nabla S}{\partial t} + z_{\alpha} \frac{\partial \nabla T}{\partial t} \right)$$

$$- \frac{1}{2} \nabla \left(\zeta^{2} \boldsymbol{U}_{\alpha} \cdot \nabla S \right) - \nabla \left(\zeta \boldsymbol{U}_{\alpha} \cdot \nabla T \right) + \nabla \left(\frac{1}{2} \zeta^{2} S^{2} \right) + \nabla \left(\zeta T S \right)$$

$$D^{\nu} = \frac{(\zeta - h)}{2} \frac{\partial \psi \zeta}{\partial t} - \frac{(\zeta^2 - \zeta h + h^2)}{6} \frac{\partial \psi}{\partial t} + \frac{\partial}{\partial t} \left\{ \psi \left(\frac{z_{\alpha}^2}{2} - \zeta z_{\alpha} \right) \right\}$$

+ $\frac{(\zeta - h)}{2} \nabla \left\{ U_{\alpha} \cdot (\psi \zeta) \right\} - \frac{(\zeta^2 - \zeta h + h^2)}{6} \nabla \left(U_{\alpha} \cdot \psi \right)$
+ $\nabla \left[U_{\alpha} \cdot \left\{ \psi \left(\frac{z_{\alpha}^2}{2} - \zeta z_{\alpha} \right) \right\} \right]$
- $\psi \left\{ \frac{(\zeta^2 + \zeta h - 2h^2) S}{6} + \frac{(\zeta + h) T}{2} \right\}$

$$\begin{aligned} \overline{\boldsymbol{\xi}} &= \left(\overline{\xi^{x}}, \overline{\xi^{y}}\right), \, \overline{\boldsymbol{\xi}^{\nu}} = \left(\overline{\xi^{\nu^{x}}}, \overline{\xi^{\nu^{y}}}\right), \\ \overline{\xi^{x}} &= -V_{\alpha} \left\{ \frac{\partial z_{\alpha}}{\partial x} \left(z_{\alpha} \frac{\partial S}{\partial y} + \frac{\partial T}{\partial y} \right) - \frac{\partial z_{\alpha}}{\partial y} \left(z_{\alpha} \frac{\partial S}{\partial x} + \frac{\partial T}{\partial x} \right) \right\} \\ &- \left(\frac{\partial V_{\alpha}}{\partial x} - \frac{\partial U_{\alpha}}{\partial y} \right) \left[\left\{ \frac{z_{\alpha}^{2}}{2} - \frac{(\zeta^{2} - \zeta h + h^{2})}{6} \right\} \frac{\partial S}{\partial y} + \left\{ z_{\alpha} - \frac{(\zeta - h)}{2} \right\} \frac{\partial T}{\partial y} \right] \end{aligned}$$

$$\overline{\xi^{y}} = U_{\alpha} \left\{ \frac{\partial z_{\alpha}}{\partial x} \left(z_{\alpha} \frac{\partial S}{\partial y} + \frac{\partial T}{\partial y} \right) - \frac{\partial z_{\alpha}}{\partial y} \left(z_{\alpha} \frac{\partial S}{\partial x} + \frac{\partial T}{\partial x} \right) \right\} \\ + \left(\frac{\partial V_{\alpha}}{\partial x} - \frac{\partial U_{\alpha}}{\partial y} \right) \left[\left\{ \frac{z_{\alpha}^{2}}{2} - \frac{(\zeta^{2} - \zeta h + h^{2})}{6} \right\} \frac{\partial S}{\partial x} + \left\{ z_{\alpha} - \frac{(\zeta - h)}{2} \right\} \frac{\partial T}{\partial x} \right]$$

$$\overline{\xi^{\nu^{x}}} = -V_{\alpha} \left[\frac{\partial}{\partial x} \left\{ \psi^{y} \left(\frac{1}{2} z_{\alpha}^{2} - z_{\alpha} \zeta \right) \right\} - \frac{(\zeta^{2} - \zeta h + h^{2})}{6} \frac{\partial \psi^{y}}{\partial x} + \frac{(\zeta - h)}{2} \frac{\partial \psi^{y} \zeta}{\partial x} - \frac{\partial}{\partial y} \left\{ \psi^{x} \left(\frac{1}{2} z_{\alpha}^{2} - z_{\alpha} \zeta \right) \right\} + \frac{(\zeta^{2} - \zeta h + h^{2})}{6} \frac{\partial \psi^{x}}{\partial y} - \frac{(\zeta - h)}{2} \frac{\partial \psi^{x} \zeta}{\partial y} \right] - \left(\frac{\partial V_{\alpha}}{\partial x} - \frac{\partial U_{\alpha}}{\partial y} \right) \psi^{y} \left\{ \frac{z_{\alpha}^{2}}{2} - z_{\alpha} \zeta + \frac{(2\zeta^{2} - 2\zeta h - h^{2})}{6} \right\}$$

$$\overline{\xi^{\nu^{y}}} = U_{\alpha} \left[\frac{\partial}{\partial x} \left\{ \psi^{y} \left(\frac{1}{2} z_{\alpha}^{2} - z_{\alpha} \zeta \right) \right\} - \frac{(\zeta^{2} - \zeta h + h^{2})}{6} \frac{\partial \psi^{y}}{\partial x} + \frac{(\zeta - h)}{2} \frac{\partial \psi^{y} \zeta}{\partial x} - \frac{\partial}{\partial y} \left\{ \psi^{x} \left(\frac{1}{2} z_{\alpha}^{2} - z_{\alpha} \zeta \right) \right\} + \frac{(\zeta^{2} - \zeta h + h^{2})}{6} \frac{\partial \psi^{x}}{\partial y} - \frac{(\zeta - h)}{2} \frac{\partial \psi^{x} \zeta}{\partial y} \right] + \left(\frac{\partial V_{\alpha}}{\partial x} - \frac{\partial U_{\alpha}}{\partial y} \right) \psi^{x} \left\{ \frac{z_{\alpha}^{2}}{2} - z_{\alpha} \zeta + \frac{(2\zeta^{2} - 2\zeta h - h^{2})}{6} \right\}$$

and $(\psi^x, \psi^y) = \psi$.

The continuity equation is obtained by integrating equation (5) with the kinematic bottom and free surface boundary conditions, giving:

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \{(\zeta + h) \boldsymbol{U}_{\alpha}\} + \mu^{2} \left(\boldsymbol{M} + \boldsymbol{M}^{\nu}\right) = O\left(\mu^{4}\right)$$
(148)

where

$$\boldsymbol{M} = -\nabla \cdot \left[\left(\zeta + h\right) \left\{ \left(\frac{\left(\zeta^2 - \zeta h + h^2\right)}{6} - \frac{z_{\alpha}^2}{2} \right) \nabla S + \left(\frac{\left(\zeta - h\right)}{2} - z_{\alpha} \right) \nabla T \right\} \right]$$
$$\boldsymbol{M}^{\nu} = \nabla \cdot \left[\boldsymbol{\psi} \left(\zeta + h\right) \left\{ \frac{z_{\alpha}^2}{2} - z_{\alpha} \zeta + \frac{\left(2\zeta^2 - 2\zeta h - h^2\right)}{6} \right\} \right]$$

15.4 Subgrid Scale Eddy Viscosity Model and Bottom Friction

It is expected that, in the horizontal plane, the depth-integrated model will be able to resolve eddy scales larger than the grid size. For subgrid scale dissipation, the Smagorinsky model is used for the horizontal eddy viscosity, and is given in non-dimensional form as

$$\nu_t^{h'} = \alpha \mu h_o \sqrt{gh_o} \left[2 \left(\frac{\partial U_\alpha}{\partial x} \right)^2 + 2 \left(\frac{\partial V_\alpha}{\partial y} \right)^2 + 2 \left(\frac{\partial W_o}{\partial z} \right)^2 + \left(\frac{\partial V_\alpha}{\partial x} + \frac{\partial U_\alpha}{\partial y} \right)^2 \right]^{1/2} + \alpha h_o \sqrt{gh_o} O\left(\mu^2\right)$$
(149)

Comparing equation (33) with equation (2), it is evident that the initial calculation of the scale for the horizontal eddy viscosity was incorrect. This scale is changed from $\alpha h_o \sqrt{gh_o}$ to $\alpha \mu h_o \sqrt{gh_o}$. The reason for this change is that, when writing equation (2), before the long

wave scaling and perturbation, it is not yet known that $\frac{\partial u}{\partial z}$ is small. The implication of this change is that the order of the leading error of the equations (31) and (32) may be regarded to be

$$O\left(\mu^4, \alpha\mu^4, \beta\mu^3\right) \tag{150}$$

and to provide a model with consistent error, α need not be considered small, or $\alpha = C_s^2 \Delta^2 = O(1)$. Note that the lengthscale, Δ , has been scaled by the water depth. This implies that, for typical values of C_s , the grid size should be less than approximately 5 times the local water depth.

To approximate the bottom stress, a quadratic friction equation is used:

$$\tau_b^x = C_f u \sqrt{u^2 + v^2} , \quad \tau_b^y = C_f v \sqrt{u^2 + v^2}$$
(151)

where the τ_b^x and τ_b^y are the bottom stresses in the x and y directions respectively and u and v are the depth averaged velocities in the x and y directions respectively. The roughness coefficient $C_f = f/4$ (Chen and Jirka, 1995) and f is estimated using the Moody diagram, which here is calculated by the explicit formula given by Haaland (1983).

15.5 Limiting Case: Non-Dispersive, Inviscid Model: $\mu^2 \ll 0$; $\nu_t^h = \nu_t^v = \tau_b = 0$

Under these assumptions, the model reduces to the standard nonlinear shallow water wave equations:

$$\frac{\partial \boldsymbol{U}_{\alpha}}{\partial t} + \boldsymbol{U}_{\alpha} \cdot \nabla \boldsymbol{U}_{\alpha} + \nabla \zeta = O\left(\mu^{2}\right)$$
(152)

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \{(\zeta + h) \, \boldsymbol{U}_{\alpha}\} = O\left(\mu^2\right) \tag{153}$$

15.6 Limiting Case: Weakly-Dispersive, Inviscid Model: $\mu^4 \ll 0$; $\nu_t^h = \nu_t^v = \tau_b = 0$

Under these assumptions, the model reduces to the extended Boussinesq equations of Chen (2006), prior to the ad-hod additions of dissipation sub-models:

$$\frac{\partial \boldsymbol{U}_{\alpha}}{\partial t} + \boldsymbol{U}_{\alpha} \cdot \nabla \boldsymbol{U}_{\alpha} + \nabla \zeta + \mu^2 \left(\boldsymbol{D} + \overline{\boldsymbol{\xi}} \right) = O\left(\mu^4\right)$$
(154)

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \{(\zeta + h) \boldsymbol{U}_{\alpha}\} + \mu^{2} \boldsymbol{M} = O\left(\mu^{4}\right)$$
(155)

It is important now to go back to the original scaling argument (see beginning of section 2.2). It was mentioned in this earlier discussion that there was no clear reason to choose either μ^2 or $\beta\mu$ as the perturbation expansion parameter. If one derives the inviscid model, given above as equations (38) and (39), the expansion parameter is clearly μ^2 . It is then reasonable

to extrapolate that, comparing the inviscid and viscous equations, the new dispersive terms appearing in the viscous equations are in fact order $\beta\mu$:

$$\frac{\partial \boldsymbol{U}_{\alpha}}{\partial t} + \boldsymbol{U}_{\alpha} \cdot \nabla \boldsymbol{U}_{\alpha} + \nabla \zeta + \mu^{2} \left(\boldsymbol{D} + \overline{\boldsymbol{\xi}}\right) + \beta \mu \left(\boldsymbol{D}^{\nu} + \overline{\boldsymbol{\xi}^{\nu}}\right)
- \alpha \mu \nabla \cdot \left(\nu_{t}^{h} \nabla \boldsymbol{U}_{\alpha}\right) + \beta \mu \nu_{t}^{v} \nabla S + \beta \mu \frac{\boldsymbol{\tau}_{b}}{\zeta + h}
= O\left(\mu^{4}, \alpha \mu^{3}, \beta \mu^{3}, \beta^{2} \mu^{2}\right)$$
(156)

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \left\{ \left(\zeta + h\right) \boldsymbol{U}_{\alpha} \right\} + \mu^{2} \boldsymbol{M} + \beta \mu \boldsymbol{M}^{\nu} = O\left(\mu^{4}, \beta^{2} \mu^{2}\right)$$
(157)

The above equations, (40) and (41), are the derived equations in their final form, and will be used for the rest of this paper. Also note that following this argument, the vertical profile of horizontal velocity is now

$$\boldsymbol{U} = \boldsymbol{U}_{\alpha} + \mu^2 \boldsymbol{U}_1^{\phi} + \beta \mu \boldsymbol{U}_1^r + O\left(\mu^4, \beta^2 \mu^2\right)$$
(158)

add the viscous contribution to the profile is evident.

15.7 Limiting Case: Non-Dispersive, Weakly-Turbulent Model: $\mu^2 \ll 0, \alpha^2/\mu = O(1), \beta^2/\mu = O(1)$

Under this set of assumptions, $O(\beta\mu)$ terms are retained as they will be greater than the truncated $O(\mu^2)$ dispersive terms:

$$\frac{\partial \boldsymbol{U}_{\alpha}}{\partial t} + \boldsymbol{U}_{\alpha} \cdot \nabla \boldsymbol{U}_{\alpha} + \nabla \zeta + \beta \mu \left(\boldsymbol{D}^{\nu} + \overline{\boldsymbol{\xi}^{\nu}} \right)
- \alpha \mu \nabla \cdot \left(\nu_{t}^{h} \nabla \boldsymbol{U}_{\alpha} \right) + \beta \mu \nu_{t}^{v} \nabla S + \beta \mu \frac{\boldsymbol{\tau}_{b}}{\zeta + h}
= O\left(\mu^{2}, \alpha \mu^{3}, \beta \mu^{3}, \beta^{2} \mu^{2} \right)$$
(159)

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \left\{ \left(\zeta + h\right) \boldsymbol{U}_{\alpha} \right\} + \beta \mu \boldsymbol{M}^{\nu} = O\left(\mu^{2}, \beta^{2} \mu^{2}\right)$$
(160)

and the horizontal velocity is

$$\boldsymbol{U} = \boldsymbol{U}_{\alpha} + \beta \mu \boldsymbol{U}_{1}^{r} + O\left(\mu^{2}, \beta^{2} \mu^{2}\right)$$
(161)

Here, the equations indicate the interesting result that, in a physically consistent context, one cannot simply append a bottom friction term onto the inviscid shallow water wave equations in an attempt to capture dissipative effects. By including a bottom stress, a number of associated terms, all of equal order to the added bottom stress, appear in both the momentum and continuity equations. It is argued that, in any shallow flow where the bottom stress plays a non-negligible role, the equation set given above as (43) and (44) is the proper model to solve. This set includes both the vertical and horizontal vorticity that is caused by the bottom stress.

15.8 Limiting Case: Weakly-Dispersive, Weakly-Turbulent Model: $\mu^4 \ll 0, \alpha^2/\mu = O(1), \beta^2/\mu = O(1)$

This is the model presented earlier, equations (40) and (41). This model includes the 2nd order frequency dispersion correction for free surface waves as well the viscous and rotational correction due to a bottom stress. While the eddy viscosity and horizontal vorticity models are simplified, a model with known physical limitations has been derived that includes the bottom friction term commonly added, in an ad-hoc manner, to the inviscid equations. Finally, it is stated that one should take care when adding such ad-hoc models; it is clear from this exercise that (1) it is not necessary to do so - the terms can be included through a consistent derivation from the viscous primitive equations - and (2) one cannot properly add the bottom friction term without also adding a number of additional terms in both the continuity and momentum equations.

15.9 Conservative Form of Boussinesq Equations

In real nature such as coastal regions, lakes or rivers, the flow motions can easily become complex. For example, because of changes in bathymetry, the flow can be changed from subcritical to supercritical and viceversa. By the way, it is well known that the primitive variable scheme or non-conservative schemes will compute shock waves with the wrong strength and thus the wrong speed of propagation (Toro, 2001).

Conservative schemes are known as a remedy, providing more accurate and stable results. In order to make the momentum equations conservative, multiply the momentum equation (40) by the total water depth and multiply the continuity equation (41) by the horizontal velocity (42). Assuming that the bottom does not vary in time ($h_t = 0$), the two multiplied equations are added, and after simple math, a set of conservative Boussinesq equations can be obtained:

$$\frac{\partial H}{\partial t} + \frac{\partial H U_{\alpha}}{\partial x} + \frac{\partial H V_{\alpha}}{\partial y} + D^c = 0$$
(162)

$$\frac{\partial HU_{\alpha}}{\partial t} + \frac{\partial HU_{\alpha}^2}{\partial x} + \frac{\partial HU_{\alpha}V_{\alpha}}{\partial y} + gH\frac{\partial\zeta}{\partial x} + gHD^x + U_{\alpha}D^c = 0$$
(163)

$$\frac{\partial HV_{\alpha}}{\partial t} + \frac{\partial HU_{\alpha}V_{\alpha}}{\partial x} + \frac{\partial HV_{\alpha}^2}{\partial y} + gH\frac{\partial\zeta}{\partial y} + gHD^y + V_{\alpha}D^c = 0$$
(164)

where $H = \zeta + h$ is a total water depth and the D^x and D^y are the 2nd order terms $(O(\mu^2, \beta \mu, \alpha \mu))$ of the depth integrated x and y horizontal momentum equations and the D^c includes the 2nd order terms of the continuity equation.

15.10 Time Integration

A standard issue for the extended Boussinesq-type equations, which include 1st to 3rd order spatial derivatives, the time integration should be fourth order accurate. This prevents numerical truncation errors of the same form as included derivatives. A third-order Adams-Bashforth predictor and the fourth-order Adams-Moulton corrector scheme are used for the time integration.

The predictor step is

$$\zeta^{n+1} = \zeta^n + \frac{\Delta t}{12} \left(23E^n - 16E^{n-1} + 5E^{n-2} \right)$$
(165)

$$P^{n+1} = P^n + \frac{\Delta t}{12} \left(23F^n - 16F^{n-1} + 5F^{n-2} \right) + 2F_1^n - 3F_1^{n-1} + F_1^{n-2} + F_v^p$$
(166)

$$Q^{n+1} = Q^n + \frac{\Delta t}{12} \left(23G^n - 16G^{n-1} + 5G^{n-2} \right) + 2G_1^n - 3G_1^{n-1} + G_1^{n-2} + G_v^p$$
(167)

where P, Q, E, F and G are defined as

$$P = H\left[U_{\alpha} + \frac{1}{2}\left(z_{\alpha}^{2} - \zeta^{2}\right)U_{\alpha x x} + (z_{\alpha} - \zeta)(hU_{\alpha})_{x x} - \zeta_{x}\left\{\zeta U_{\alpha x} + (hU_{\alpha})_{x}\right\}\right]$$
(168)

$$Q = H\left[V_{\alpha} + \frac{1}{2}\left(z_{\alpha}^{2} - \zeta^{2}\right)V_{\alpha yy} + (z_{\alpha} - \zeta)\left(hV_{\alpha}\right)_{yy} - \zeta_{y}\left\{\zeta V_{\alpha y} + (hV_{\alpha})_{y}\right\}\right]$$
(169)

$$E = E_{LO} + E_D + E_V = 0 (170)$$

$$F = F_{LO} + F_D \tag{171}$$

$$G = G_{LO} + G_D \tag{172}$$

 E_{LO}, F_{LO} , and G_{LO} are rewritten by

$$E_{LO} = \frac{\partial H U_{\alpha}}{\partial x} + \frac{\partial H V_{\alpha}}{\partial y} \tag{173}$$

$$F_{LO} = \frac{\partial}{\partial x} \left(HU_{\alpha}^2 + \frac{1}{2}gH^2 \right) + \frac{\partial HU_{\alpha}V_{\alpha}}{\partial y} - gH\frac{\partial h}{\partial x}$$
(174)

$$G_{LO} = \frac{\partial H U_{\alpha} V_{\alpha}}{\partial x} + \frac{\partial}{\partial y} \left(H V_{\alpha}^2 + \frac{1}{2} g H^2 \right) - g H \frac{\partial h}{\partial y}$$
(175)

and E_D , E_V , F_D , G_D , F_1 and G_1 are defined as

$$E_D = \left[H\left\{ \left(\frac{1}{6} \left(\zeta^2 - \zeta h + h^2 \right) - \frac{1}{2} z_\alpha^2 \right) \nabla S + \left(\frac{1}{2} \left(\zeta - h \right) - z_\alpha \right) \nabla T \right\} \right]_x + \left[H\left\{ \left(\frac{1}{6} \left(\zeta^2 - \zeta h + h^2 \right) - \frac{1}{2} z_\alpha^2 \right) \nabla S + \left(\frac{1}{2} \left(\zeta - h \right) - z_\alpha \right) \nabla T \right\} \right]_y$$
(176)

$$E_{V} = - \left[H\psi^{x} \left\{ \frac{\zeta^{2}}{2} - z_{\alpha}\zeta + \frac{(2\zeta^{2} - 2\zeta h - h^{2})}{6} \right\} \right]_{x} - \left[H\psi^{y} \left\{ \frac{\zeta^{2}}{2} - z_{\alpha}\zeta + \frac{(2\zeta^{2} - 2\zeta h - h^{2})}{6} \right\} \right]_{y}$$
(177)

$$(F_{D}, G_{D}) = H \left[\frac{1}{2} \nabla \left(\zeta^{2} \boldsymbol{U}_{\alpha} \cdot \nabla S \right) + \nabla \left(\zeta \boldsymbol{U}_{\alpha} \cdot \nabla T \right) - \frac{1}{2} \nabla \left(\zeta^{2} S^{2} \right) \right] - \frac{1}{2} \nabla \left(z_{\alpha}^{2} \boldsymbol{U}_{\alpha} \cdot \nabla S \right) - \nabla \left(z_{\alpha} \boldsymbol{U}_{\alpha} \cdot \nabla T \right) - \left(T \nabla T \right) - \nabla \left(\zeta T S \right) \right] - \frac{\left(\zeta - h \right)}{2} \nabla \left\{ \boldsymbol{U}_{\alpha} \cdot \left(\boldsymbol{\psi} \zeta \right) \right\} + \frac{\left(\zeta^{2} - \zeta h + h^{2} \right)}{6} \nabla \left(\boldsymbol{U}_{\alpha} \cdot \boldsymbol{\psi} \right) \right] - \nabla \left[\boldsymbol{U}_{\alpha} \cdot \left\{ \boldsymbol{\psi} \left(\frac{z_{\alpha}^{2}}{2} - \zeta z_{\alpha} \right) \right\} \right] + \left\{ \boldsymbol{\psi} \left\{ \frac{\left(\zeta^{2} + \zeta h - 2h^{2} \right) S}{6} + \frac{HT}{2} \right\} - \overline{\boldsymbol{\xi}} \right\} + \nabla \cdot \left(\nu_{t}^{h} \nabla \boldsymbol{U}_{\alpha} \right) - \nu_{t}^{v} \nabla S - \frac{\tau_{b}}{\rho H} \right]$$
(178)

$$F_{1} = \frac{H}{2} \left(\zeta^{2} - z_{\alpha}^{2} \right) v_{xy} - H \left(z_{\alpha} - \zeta \right) \left(hv \right)_{xy} + H \zeta_{x} \left\{ \zeta v_{y} + \left(hv \right)_{y} \right\}$$
(179)

$$G_{1} = \frac{H}{2} \left(\zeta^{2} - z_{\alpha}^{2} \right) u_{xy} - H \left(z_{\alpha} - \zeta \right) \left(hu \right)_{xy} + H \zeta_{y} \left\{ \zeta u_{x} + \left(hu \right)_{x} \right\}$$
(180)

 F_v^p and G_v^p are rewritten by

$$F_{v}^{p} = \frac{H^{n} \left(\zeta^{2} - \zeta h + h^{2} + 3z_{\alpha}^{2}\right)^{n}}{6} \left\{ 2 \left(\psi^{x}\right)^{n} - 3 \left(\psi^{x}\right)^{n-1} + \left(\psi^{x}\right)^{n-2} \right\} - \frac{H^{n} \left(\zeta - h - 2z_{\alpha}\right)^{n}}{2} \left\{ 2 \left(\psi^{x}\zeta\right)^{n} - 3 \left(\psi^{x}\zeta\right)^{n-1} + \left(\psi^{x}\zeta\right)^{n-2} \right\}$$
(181)

$$G_{v}^{p} = \frac{H^{n} \left(\zeta^{2} - \zeta h + h^{2} + 3z_{\alpha}^{2}\right)^{n}}{6} \left\{ 2 \left(\psi^{y}\right)^{n} - 3 \left(\psi^{y}\right)^{n-1} + \left(\psi^{y}\right)^{n-2} \right\} - \frac{H^{n} \left(\zeta - h - 2z_{\alpha}\right)^{n}}{2} \left\{ 2 \left(\psi^{y}\zeta\right)^{n} - 3 \left(\psi^{y}\zeta\right)^{n-1} + \left(\psi^{y}\zeta\right)^{n-2} \right\}$$
(182)

The corrector step is

$$\zeta^{n+1} = \zeta^n + \frac{\Delta t}{24} \left(9E^{n+1} + 19E^n - 5E^{n-1} + E^{n-2}\right)$$
(183)

$$U^{n+1} = U^n + \frac{\Delta t}{24} \left(9F^{n+1} + 19F^n - 5F^{n-1} + F^{n-2}\right) + F_1^{n+1} - F_1^n + F_v^c$$
(184)

$$V^{n+1} = V^n + \frac{\Delta t}{24} \left(9G^{n+1} + 19G^n - 5G^{n-1} + G^{n-2}\right) + G^{n+1}_1 - G^n_1 + G^c_v$$
(185)

where F_v^c and G_v^c are rewritten as

$$F_{v}^{c} = \frac{H^{n+1} \left(\zeta^{2} - \zeta h + h^{2} + 3z_{\alpha}^{2}\right)^{n+1}}{6} \left\{ \left(\psi^{x}\right)^{n+1} - \left(\psi^{x}\right)^{n} \right\} - \frac{H^{n+1} \left(\zeta - h - 2z_{\alpha}\right)^{n+1}}{2} \left\{ \left(\psi^{x}\zeta\right)^{n+1} - \left(\psi^{x}\zeta\right)^{n} \right\}$$
(186)

$$G_{v}^{c} = \frac{H^{n+1} \left(\zeta^{2} - \zeta h + h^{2} + 3z_{\alpha}^{2}\right)^{n+1}}{6} \left\{ \left(\psi^{y}\right)^{n+1} - \left(\psi^{y}\right)^{n} \right\} - \frac{H^{n+1} \left(\zeta - h - 2z_{\alpha}\right)^{n+1}}{2} \left\{ \left(\psi^{y}\zeta\right)^{n+1} - \left(\psi^{y}\zeta\right)^{n} \right\}$$
(187)

After each predictor and corrector step, P and Q are solved by a matrix solver. Note that the governing equations are solved by cell averaged finite volume method, so all computed values are cell averaged values. The P and Q can be expressed as

$$P = \frac{H}{\Delta x} \left[\int_{x_{i-1/2}}^{x_{i+1/2}} U_{\alpha}(x) dx + \frac{1}{2} \left(z_{\alpha}^{2} - \zeta^{2} \right) \int_{x_{i-1/2}}^{x_{i+1/2}} U_{\alpha}(x)_{xx} dx + (z_{\alpha} - \zeta) \int_{x_{i-1/2}}^{x_{i+1/2}} \{ h U_{\alpha}(x) \}_{xx} dx - \zeta_{x} \int_{x_{i-1/2}}^{x_{i+1/2}} U_{\alpha}(x)_{x} dx - \zeta_{x} \int_{x_{i-1/2}}^{x_{i+1/2}} \{ h U_{\alpha}(x) \}_{x} dx \right]$$
(188)

$$Q = \frac{H}{\Delta y} \left[\int_{y_{j-1/2}}^{y_{j+1/2}} V_{\alpha}(y) dy + \frac{1}{2} \left(z_{\alpha}^{2} - \zeta^{2} \right) \int_{y_{j-1/2}}^{y_{j+1/2}} V_{\alpha}(y)_{yy} dy + (z_{\alpha} - \zeta) \int_{y_{j-1/2}}^{y_{j+1/2}} \{ hV_{\alpha}(y) \}_{yy} dy - \zeta_{y} \zeta \int_{y_{j-1/2}}^{y_{j+1/2}} V_{\alpha}(y)_{y} dy - \zeta_{y} \int_{y_{j-1/2}}^{y_{j+1/2}} \{ hV_{\alpha}(y) \}_{y} dy \right]$$
(189)

The equations (63) and (64) yield a diagonal matrix and can be solved efficiently. In this study, a bandwidth of three matrix is used. For x direction,

$$\alpha U_{\alpha}^{i-1} + \beta U_{\alpha}^{i} + \gamma U_{\alpha}^{i+1} = P \tag{190}$$

where

$$\alpha = H_i \left\{ \frac{z_\alpha^2 - \zeta^2}{2\Delta x^2} + \frac{(z_\alpha - \zeta)h_{i-1}}{\Delta x^2} + \frac{\zeta_x \zeta}{2\Delta x} + \frac{\zeta_x h_{i-1}}{2\Delta x} \right\}$$
(191)

$$\beta = H_i \left\{ 1 - \frac{z_\alpha^2 - \zeta^2}{\Delta x^2} + \frac{2(z_\alpha - \zeta)h_i}{\Delta x^2} \right\}$$
(192)

$$\gamma = H_i \left\{ \frac{z_\alpha^2 - \zeta^2}{2\Delta x^2} + \frac{(z_\alpha - \zeta)h_{i+1}}{\Delta x^2} - \frac{\zeta_x \zeta}{2\Delta x} - \frac{\zeta_x h_{i+1}}{2\Delta x} \right\}$$
(193)

For y direction, a straightforward procedure is used. The convergence error is defined as $\epsilon = \sum |f^{n+1} - f^{n+1}_*| / \sum |f^{n+1}|$ and it is required $\epsilon < 10^{-4}$ in order to be converged in the verifications of this study.

15.11 4th-Order Accuracy Compact MUSCL TVD Scheme for Leading Order Terms

For the analysis of F_{LO} and G_{LO} terms except the bottom slope terms gHh_x and gHh_y , a fourth-order compact MUSCL TVD scheme (Yamamoto and Daiguji, 1993) is used to construct the interface values as followings

$$\phi_{i+1/2}^{L} = \phi_i + \frac{1}{6} \left\{ \Delta^* \overline{\phi}_{i-1/2} + 2\Delta^* \widetilde{\phi}_{i+1/2} \right\}$$
(194)

$$\phi_{i+1/2}^R = \phi_{i+1} - \frac{1}{6} \left\{ 2\Delta^* \overline{\phi}_{i+1/2} + 2\Delta^* \widetilde{\phi}_{i+3/2} \right\}$$
(195)

where

$$\Delta^* \overline{\phi}_{i-1/2} = minmod \left(\Delta^* \phi_{i-1/2}, b \Delta^* \phi_{i+1/2} \right)$$
(196)

$$\Delta^* \widetilde{\phi}_{i+1/2} = minmod \left(\Delta^* \phi_{i+1/2}, b \Delta^* \phi_{i-1/2} \right)$$
(197)

$$\Delta^* \overline{\phi}_{i+1/2} = minmod \left(\Delta^* \phi_{i+1/2}, b \Delta^* \phi_{i+3/2} \right)$$
(198)

$$\Delta^* \widetilde{\phi}_{i+3/2} = minmod \left(\Delta^* \phi_{i+3/2}, b \Delta^* \phi_{i+1/2} \right)$$
(199)

$$\Delta^* \phi_{i+1/2} = \Delta \phi_{i+1/2} - \frac{1}{6} \Delta^3 \overline{\phi}_{i+1/2}$$
(200)

$$\Delta^{3}\overline{\phi}_{i+1/2} = \Delta\overline{\phi}_{i-1/2} - 2\Delta\overline{\phi}_{i+1/2} + \Delta\overline{\phi}_{i+3/2}$$
(201)

$$\Delta \overline{\phi}_{i-1/2} = minmod \left(\Delta \phi_{i-1/2}, b_1 \Delta \phi_{i+1/2}, b_1 \Delta \phi_{i+3/2} \right)$$
(202)

$$\Delta \overline{\phi}_{i+1/2} = minmod \left(\Delta \phi_{i+1/2}, b_1 \Delta \phi_{i+3/2}, b_1 \Delta \phi_{i-1/2} \right)$$
(203)

$$\Delta \overline{\phi}_{i+3/2} = minmod \left(\Delta \phi_{i+3/2}, b_1 \Delta \phi_{i-1/2}, b_1 \Delta \phi_{i+1/2} \right)$$
(204)

$$minmod(i,j) = sign(i) \max\left[0, \min\left\{|i|, b_1 sign(i)\right\}\right]$$
(205)

$$ninmod(i, j, k) = sign(i) \max\left[0, \min\left\{|i|, b_1 sign(i), b_1 sign(i)k\right\}\right]$$

$$(206)$$

in which the coefficients $b_1 = 2$ and $1 < b \leq 4$ and more details of this numerical scheme are described in Yamamoto and Daiguji (1993). By using the constructed interface values, the numerical fluxes are computed by approximate Riemann solvers such as HLL or HLLC (Toro, 1999).

However, this numerical scheme which combines the Riemann solvers and MUSCL scheme can occur unphysical oscillations when it is applied to solve the shallow water equations system on irregular bathymetry. One of the remedy is the Surface Gradient Method (Zhou et al. 2001), which can get rid of the oscillations but the bathymetry should vary continuously. Therefore, if there are discontinuous bottom area, modified versions of the surface gradient method should be used (Zhou et al. 2002 and Kim et al. 2008). In this study, a modified version by Kim et al. (2008) is used for the simplicity of the method. Originally, the modified version of the surface gradient method was developed and verified for the combinations of HLL/HLLC and second-order MUSCL scheme. Although the results is not printed in this paper, the authors verified that it still work for the fourth-order accuracy MUSCL scheme.

15.12 Higher Order Accuracy Finite Volume Discretization for Dispersive Terms

A cell averaged value $\bar{\phi}_i$ is defined as

1

$$\bar{\phi}_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \phi(x) dx \tag{207}$$

Substituting the cell averaged value into the Taylor series $\phi = \phi_{i+1/2} + x\phi'_{i+1/2} + x^2/2\phi''_{i+1/2} + x^3/6\phi'''_{i+1/2} + x^4/24\phi''''_{i+1/2} + \cdots$ then we can express the cell averaged value with the values defined at cell interfaces (Lacor et al., 2004). For example,

$$\bar{\phi}_i = \phi_{i+1/2} - \frac{\Delta x}{2} \phi'_{i+1/2} + \frac{\Delta x^2}{6} \phi''_{i+1/2} - \frac{\Delta x^3}{24} \phi'''_{i+1/2} + \dots$$
(208)

where subscript i means the index of a cell and i + 1/2 is the index of a cell interface. By the combinations of the Taylor series expansions, the following discretization equations can be derived and used for the discretization of the dispersion terms.

$$\phi_{i+1/2} = \frac{7\left(\bar{\phi}_{i+1} + \bar{\phi}_i\right) - \left(\bar{\phi}_{i+2} + \bar{\phi}_{i-1}\right)}{12} + O\left(\Delta x^4\right) \tag{209}$$

$$\phi_{i+1/2}' = \frac{15\left(\bar{\phi}_{i+1} - \bar{\phi}_i\right) - \left(\bar{\phi}_{i+2} - \bar{\phi}_{i-1}\right)}{12\Delta x} + O\left(\Delta x^4\right) \tag{210}$$

$$\phi_{i+1/2}'' = \frac{\left(\bar{\phi}_{i+2} + \bar{\phi}_{i-1}\right) - \left(\bar{\phi}_{i+1} + \bar{\phi}_{i}\right)}{2\Delta x^2} + O\left(\Delta x^2\right)$$
(211)

16 Appendix D. Research Papers by the Model Developers

Kim, D.-H., Lynett, P. and Socolofosky, S. (2008). "A Depth-Integrated Model for Weakly Dispersive, Turbulent, and Rotational Fluid Flows." submitted to Ocean Modeling.

Augustin, L. N., Irish, J. L., and Lynett, P. "Laboratory and Numerical Studies of Wave Damping by Emergent and Near-Emergent Wetland Vegetation." submitted to Coastal Engineering, 2008.

Sitanggang, K. and Lynett, P. "Multi-scale Simulation with a Hybrid Boussinesq-RANS Hydrodynamic Model." submitted to Journal of Computational Physics, 2008.

Geist, E., Lynett, P., and Chaytor, J. "Hydrodynamic Modeling of Tsunamis from the Currituck Landslide." submitted to Marine Geology, 2008

Huntington, K., Bourgeois, J., Gelfenbaum, G., Lynett, P., Jaffe, B., Yeh, H., and Weiss, R. "Sandy Signs of Tsunami Onshore Depth and Speed." to appear in AGU EOS, 2007.

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Goff, J., Liu, P., Lynett, P., Fernando, J., Jaffe, B., Fritz, H., Higman, B., Morton, R., and Synolakis C., "Runup and Inundation Limits in Sri Lanka from the December 26th Tsunami," Earthquake Spectra, 2006, v. 22(S3), p. 155-172. 2006; similar but more detailed paper in Bulletin for the New Zealand Society for Earthquake Engineering, 2006.

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