Benchmark problem 3

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Theory employed

The models

- **Eulerian FDM for standard Boussinesq equations**
  Nonlinear code without particular run-up feature
  Used only in linear mode, with and without dispersion terms
- **Lagrangian FDM for Boussinesq type equations**
  Method particularly designed for runup
  Fully nonlinear, but weakly dispersive. Employed in hydrostatic and dispersive mode

Definitions

- h equilibrium depth, $h' = \partial h(x,t)/\partial x$ etc.
- η surface elevation
- $H = h + \eta$ total depth
- $\overline{u}$ depth-averaged horizontal velocity

Averaged material derivate

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}$$

The fully nonlinear Boussinesq equations

$$\frac{\mathrm{D}H}{\mathrm{D}t} = -H\frac{\partial \overline{u}}{\partial x},$$

$$\left(1 - \tfrac{1}{2}Hh'' - h'\frac{\partial \eta}{\partial x}\right)\frac{\mathrm{D}\overline{u}}{\mathrm{D}t} =$$

$$-g\frac{\partial \eta}{\partial x} - \frac{1}{3}H\frac{\partial}{\partial x}\left(\frac{\mathrm{D}^2 H}{\mathrm{D}t^2}\right) - \frac{2}{3}\frac{\partial H}{\partial x}\frac{\mathrm{D}^2 H}{\mathrm{D}t^2}$$

$$+\frac{h'}{H}\left(\frac{\mathrm{D}H}{\mathrm{D}t}\right)^2 - h''\overline{u}\frac{\mathrm{D}H}{\mathrm{D}t} + \left(\frac{\partial \eta}{\partial x}h'' + \frac{1}{2}Hh'''\right)u^2 + S_1 + S$$

Red: “Important” dispersion terms

Green: fully nonlinear version, $S_1 + S_2$: source terms.

Long waves and Lagrangian coordinates

Simple flow structure $\Rightarrow$ material fluid columns remain (nearly) vertical

Lagrangian coordinate a: $\frac{\mathrm{D}a}{\mathrm{D}t} = 0, \quad a(x, 0) = x$

Transformation $\frac{\mathrm{D}}{\mathrm{D}t} \to \frac{\partial}{\partial t}$, $\frac{\partial}{\partial x} \to \frac{H}{H_0}\frac{\partial}{\partial a}$ etc., $H_0 = H(a, 0)$

$$H\frac{\partial x}{\partial a} = H_0,$$

$$(1 - () - ())\frac{\partial \overline{u}}{\partial t} = -g\frac{H}{H_0}\frac{\partial H}{\partial a} + g\frac{\partial h}{\partial x} - () + () + S_1 + S_2,$$

$$\frac{\partial x}{\partial t} = \overline{u},$$

Shoreline: $H = 0$ at fixed a

The slide representation

Momentum conservation

$$(1 - () - ())\,\frac{\partial \overline{u}}{\partial t} = -g\frac{H}{H_0}\frac{\partial H}{\partial a} + g\frac{\partial h}{\partial x} - () + () + S_1 + S_2,$$

dominant explicit forcing term: $g\frac{\partial h}{\partial x}$

Additional h.o. (dispersive) terms from slide

$$S_1 + S_2 = \frac{1}{2}H\frac{\partial^3 h}{\partial t^2 \partial x} - \frac{1}{g}\left(2\overline{u}\frac{\partial^2 h}{\partial t \partial x} + \frac{\partial^2 h}{\partial t^2}\right)\frac{\partial \overline{u}}{\partial t}$$
$$- \frac{\partial^2 h}{\partial t \partial x}\frac{\partial H}{\partial t} + H\overline{u}\frac{\partial^3 h}{\partial t \partial x^2}$$

The Lagrangian Boussinesq model

- Second order FDM model. Staggered grid in space and time.
- $\boldsymbol{H}$-node at shoreline position (fixed $\boldsymbol{a}$).
  Condition $\boldsymbol{H \equiv h + \eta = 0}$ implemented directly.
- Hydrostatic (NLSW) version is explicit.
- Dispersive version implicit – iteration on nonlinearity (may be avoided)
- Wave paddle is also simple to implement.
- No smoothing or filtering for non-breaking waves.
- Hydrostatic (NLSW) extensions to non-planar waves exist

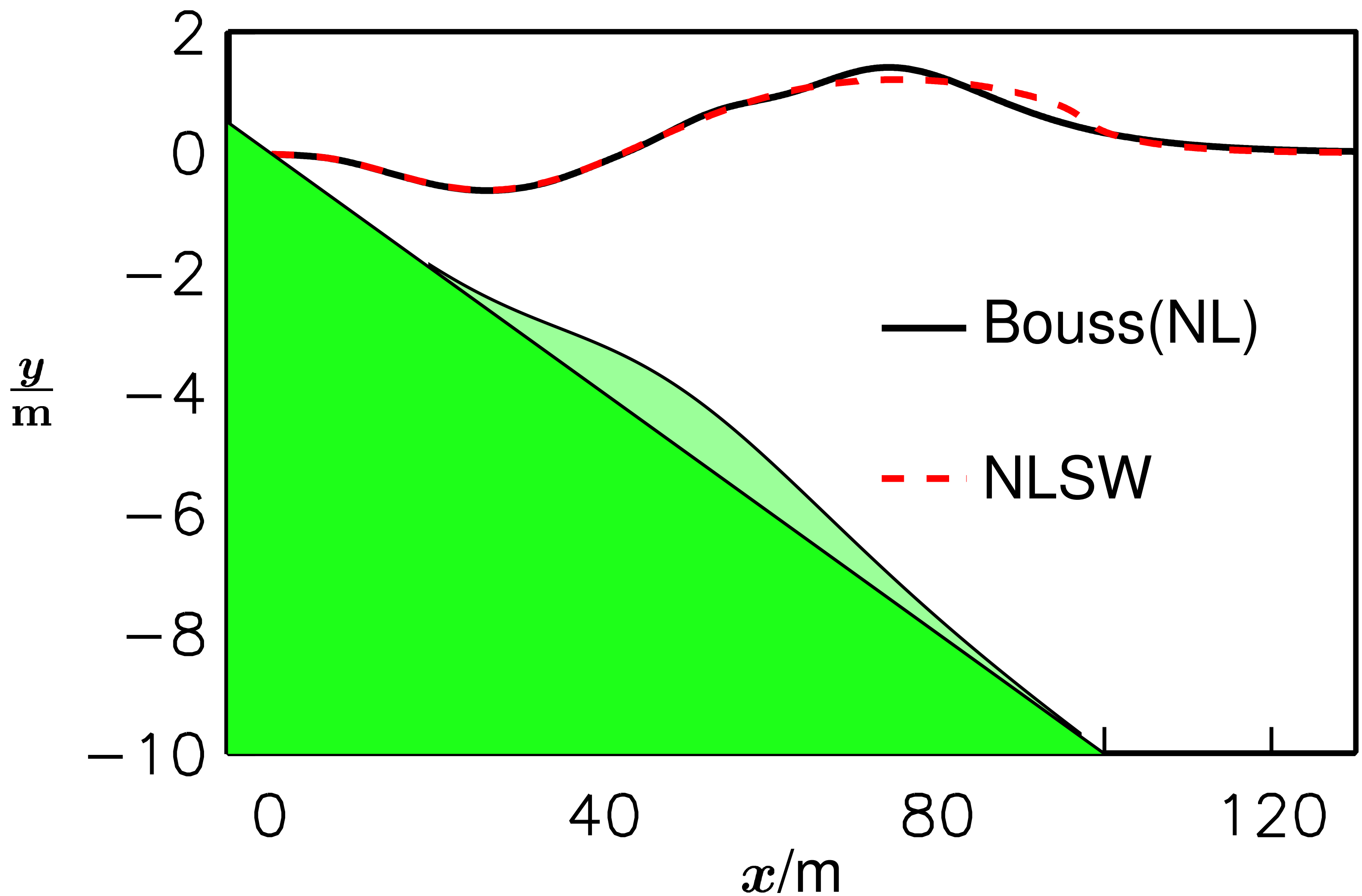
Results

Nomenclature

- ref.: (Semi-)analytic solutions from reference papers
- Bouss(NL) : Nonlinear Boussinesq solution (Lagrangian)
- NLSW: Nonlinear shallow water solution (Lagrangian)
- Bouss(L): Linear Boussinesq solution (Eulerian)
- SW(L): Linear shallow water solution (Eulerian)
- pot(NL): Full potential theory (boundary integral method)

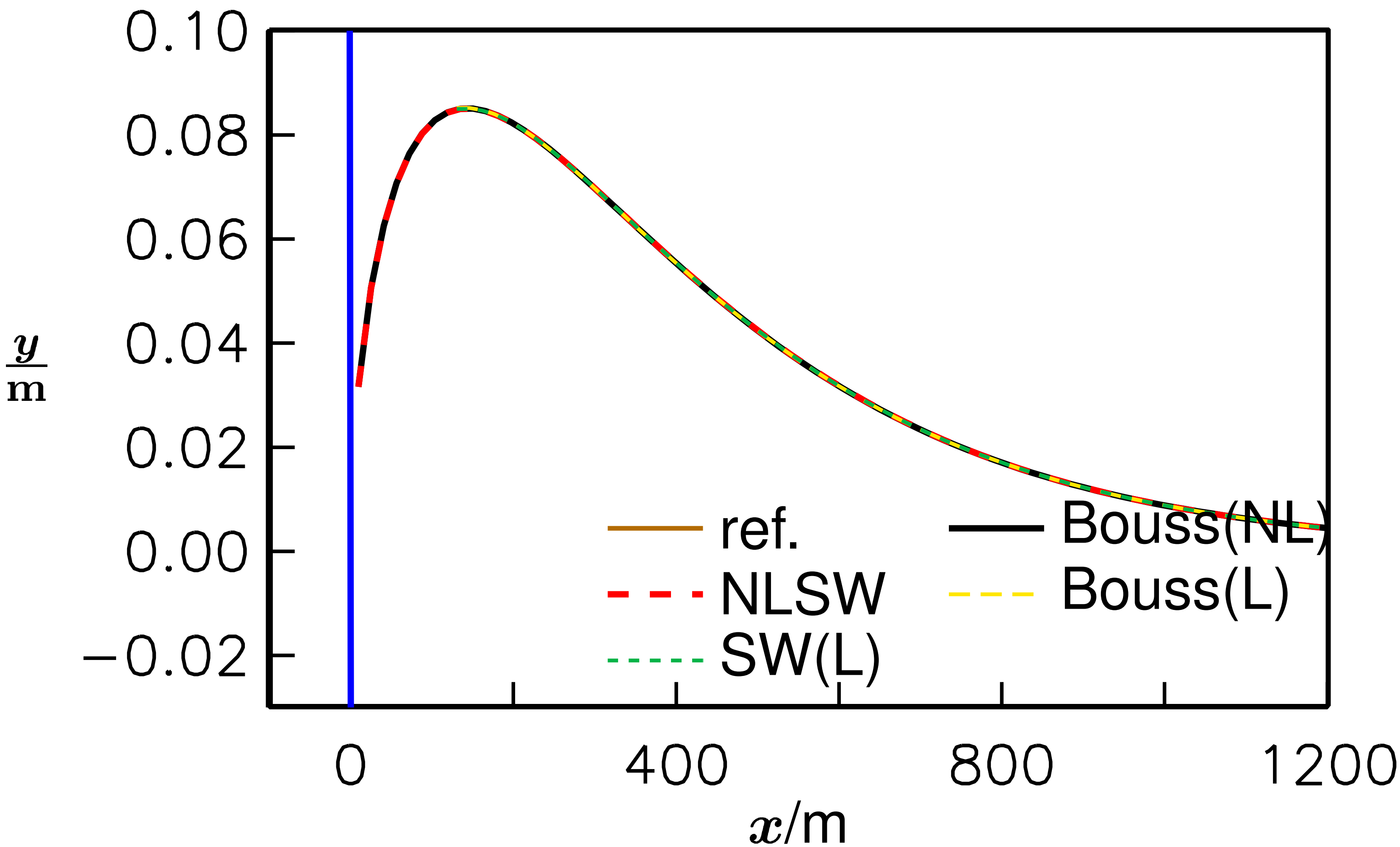
Color codes are preferably kept consistent.

Case B, $t = 4.5$



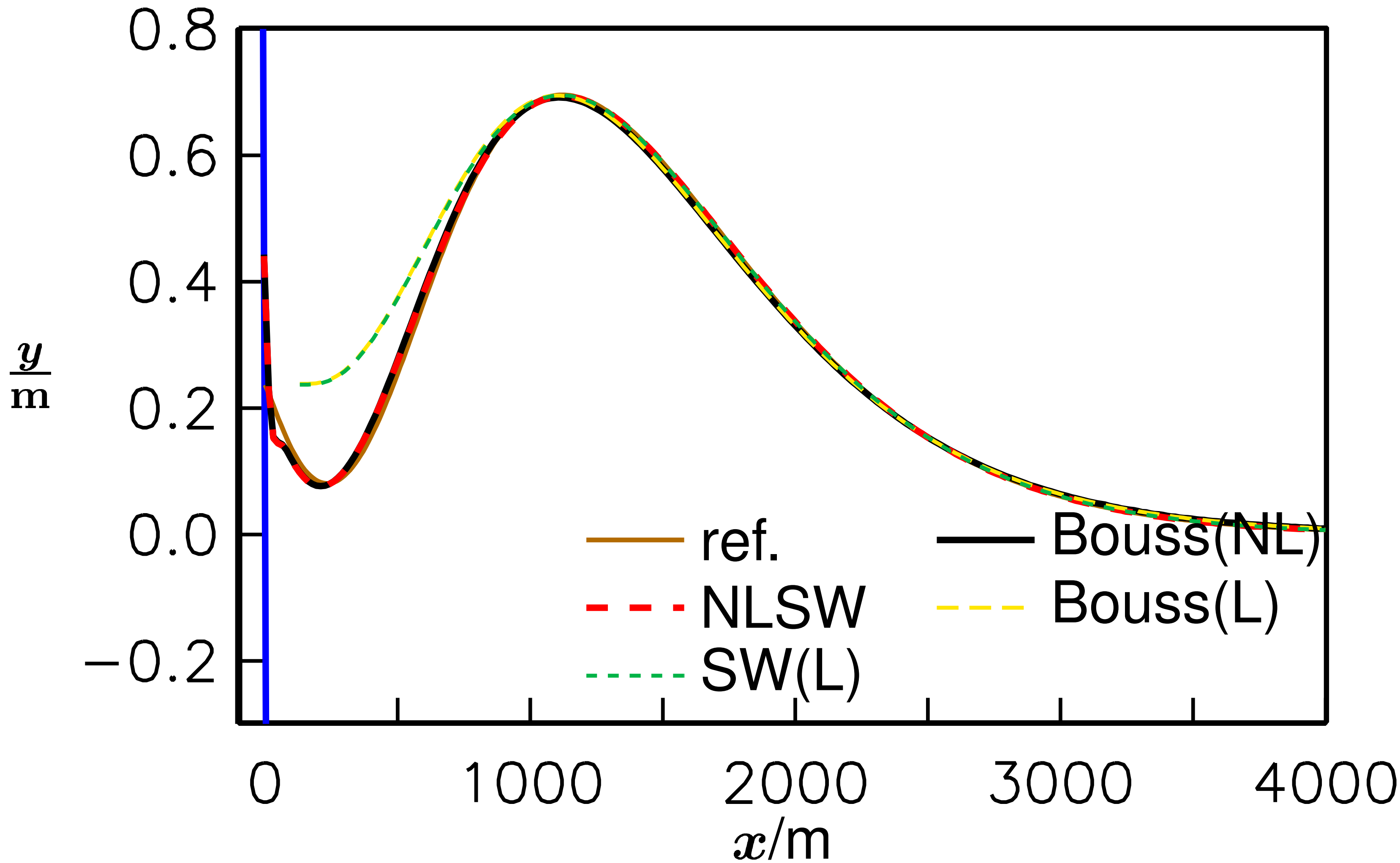
B at $t = 4.5$. Comparison of models.

Case A, $t = 0.1$



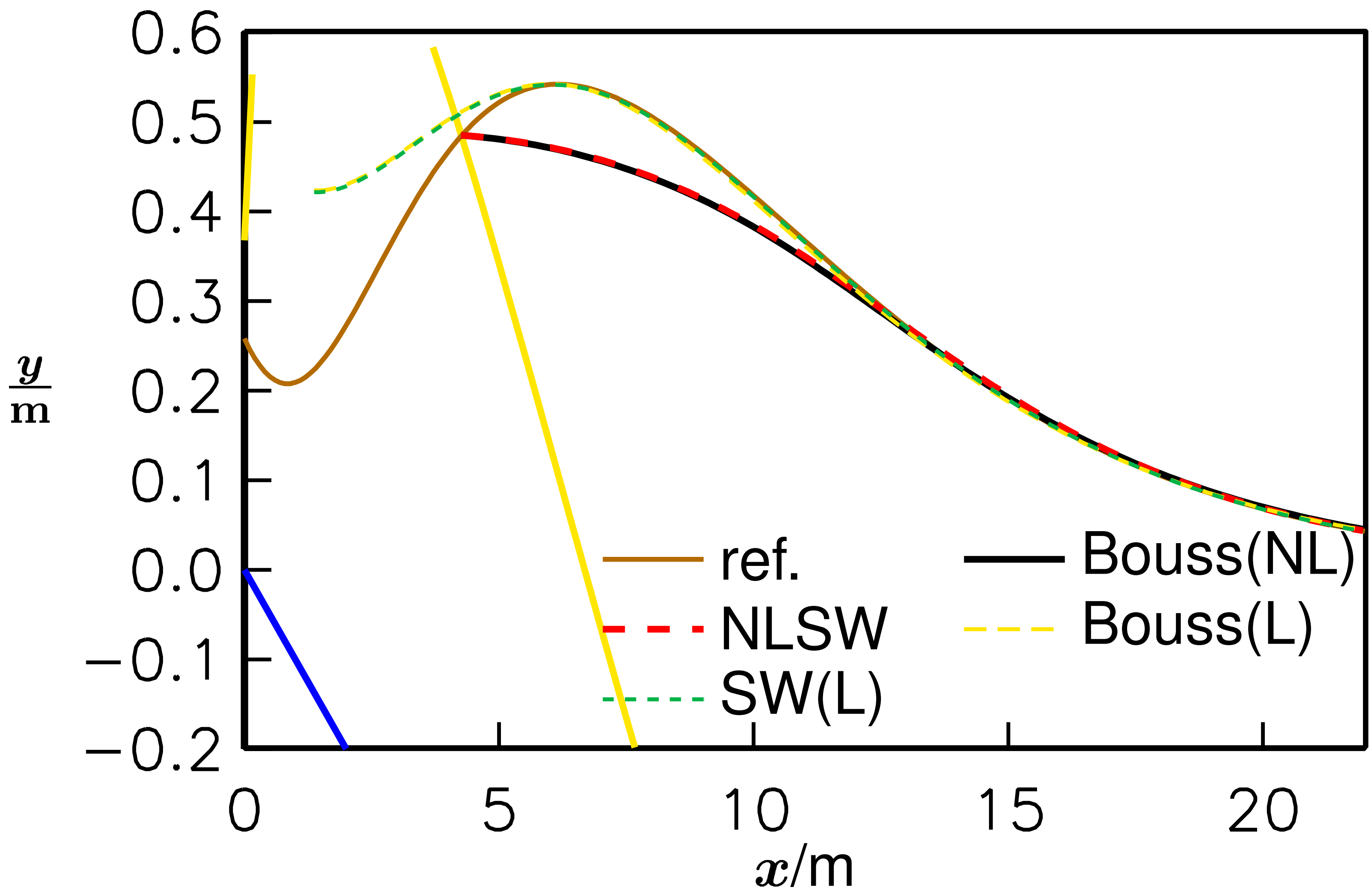
A at $t = \mathbf{0.1}$. Comparison of models.

Case A, $t = 1.5$



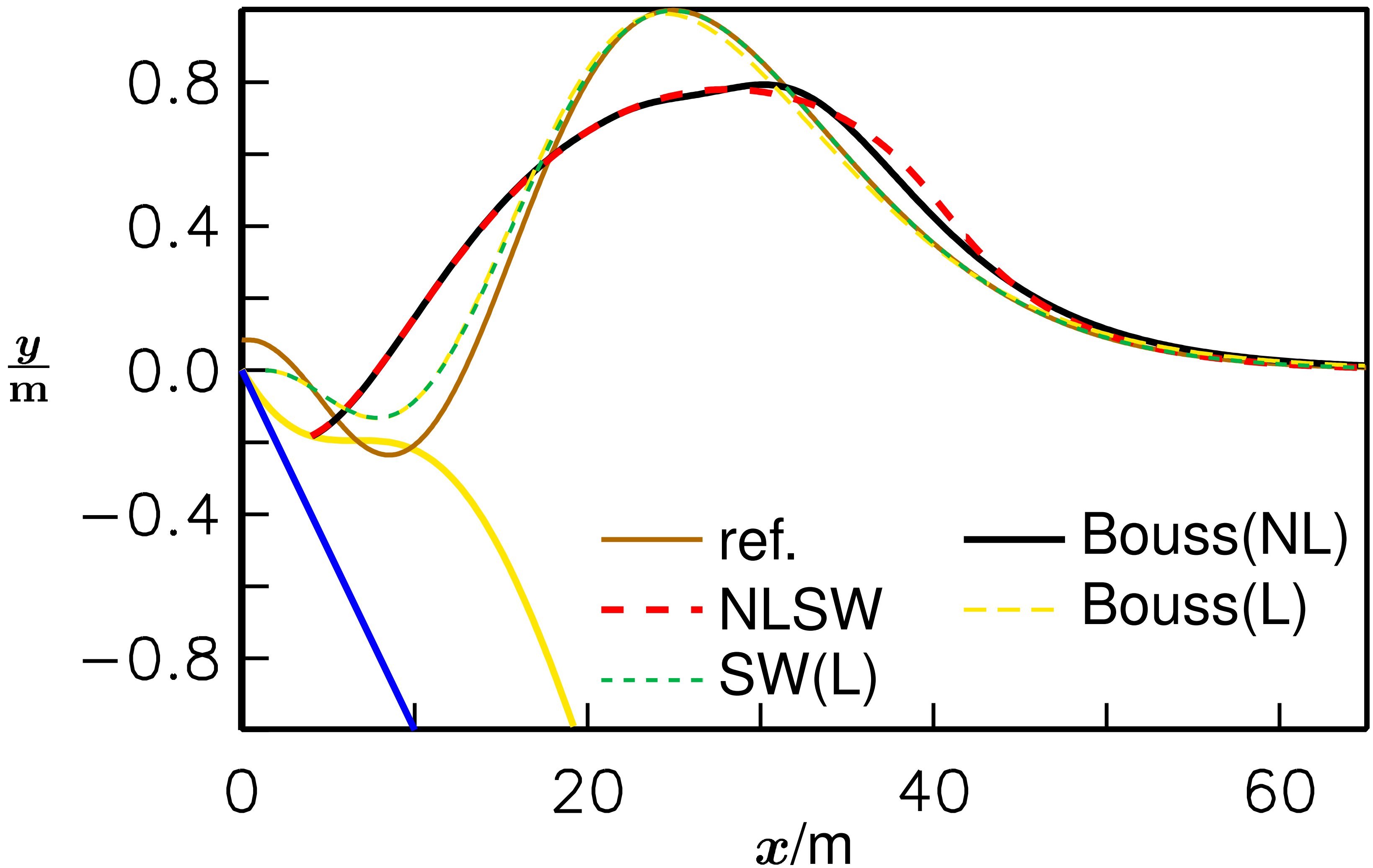
A at $t = 1.5$. Comparison of models.

Case B, $t = 1.0$



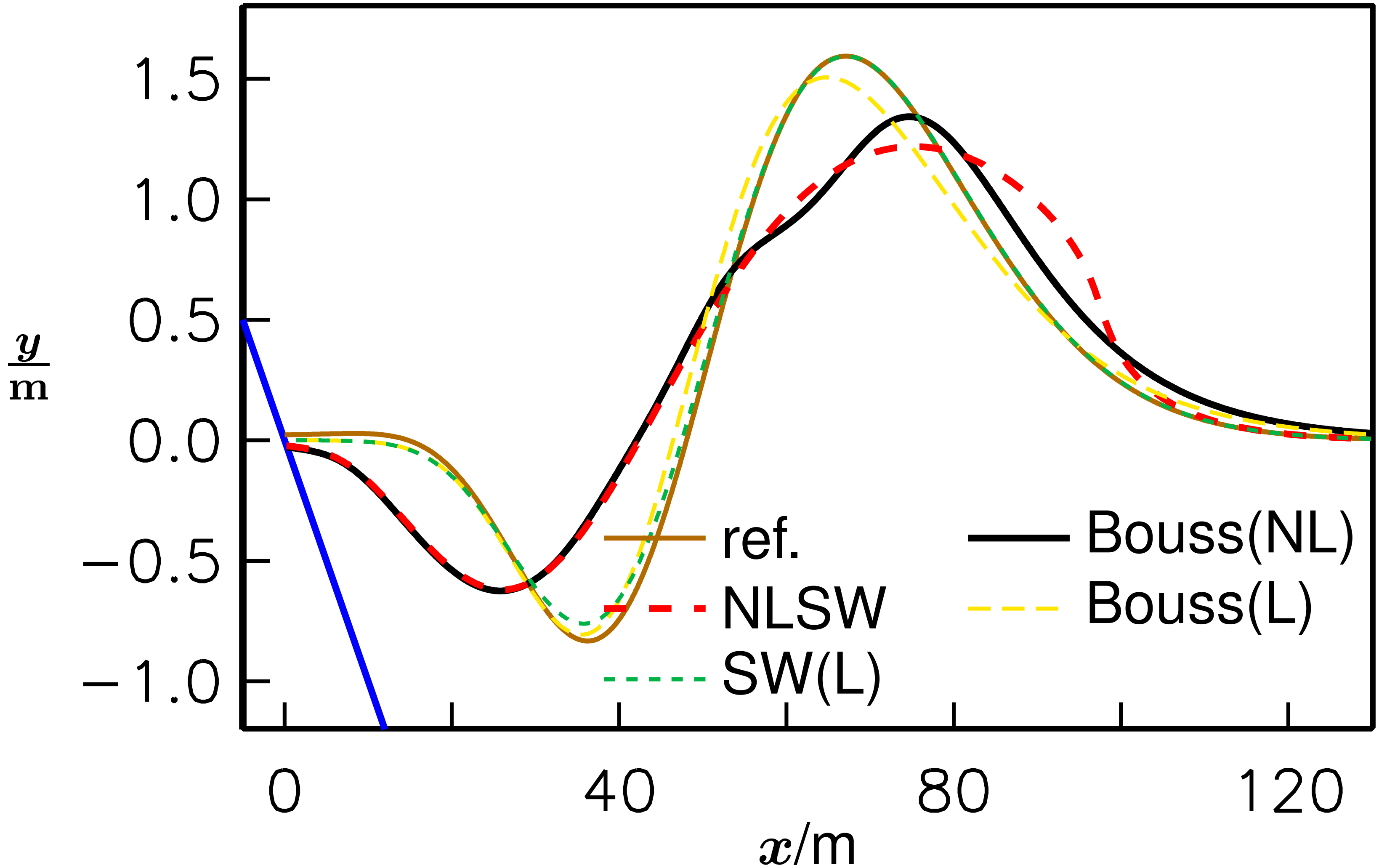
B at $t = 1.0$. Comparison of models.

Case B, $t = 2.5$



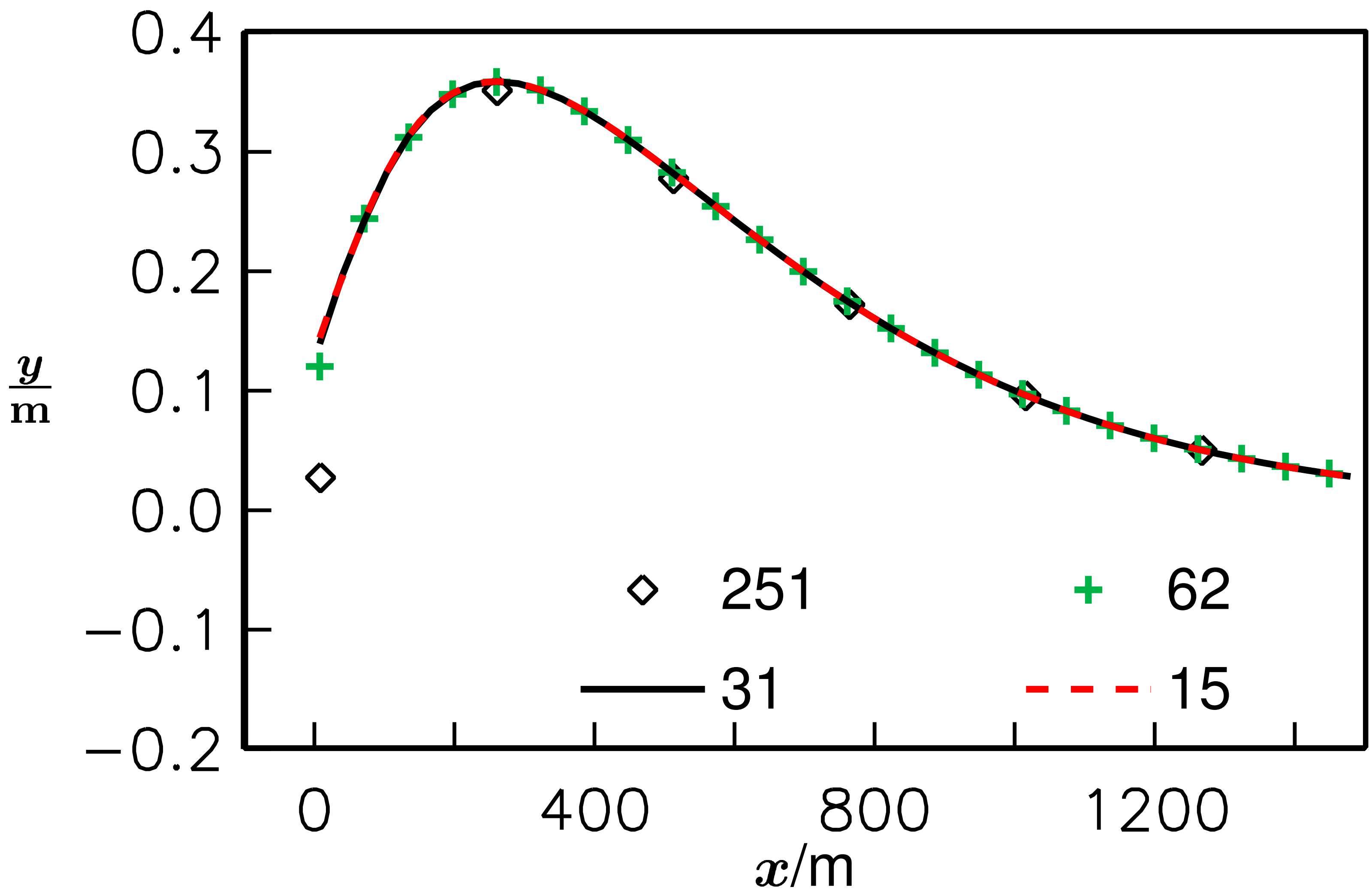
B at $t = 2.5$. Comparison of models.

Case B, $t = 4.5$



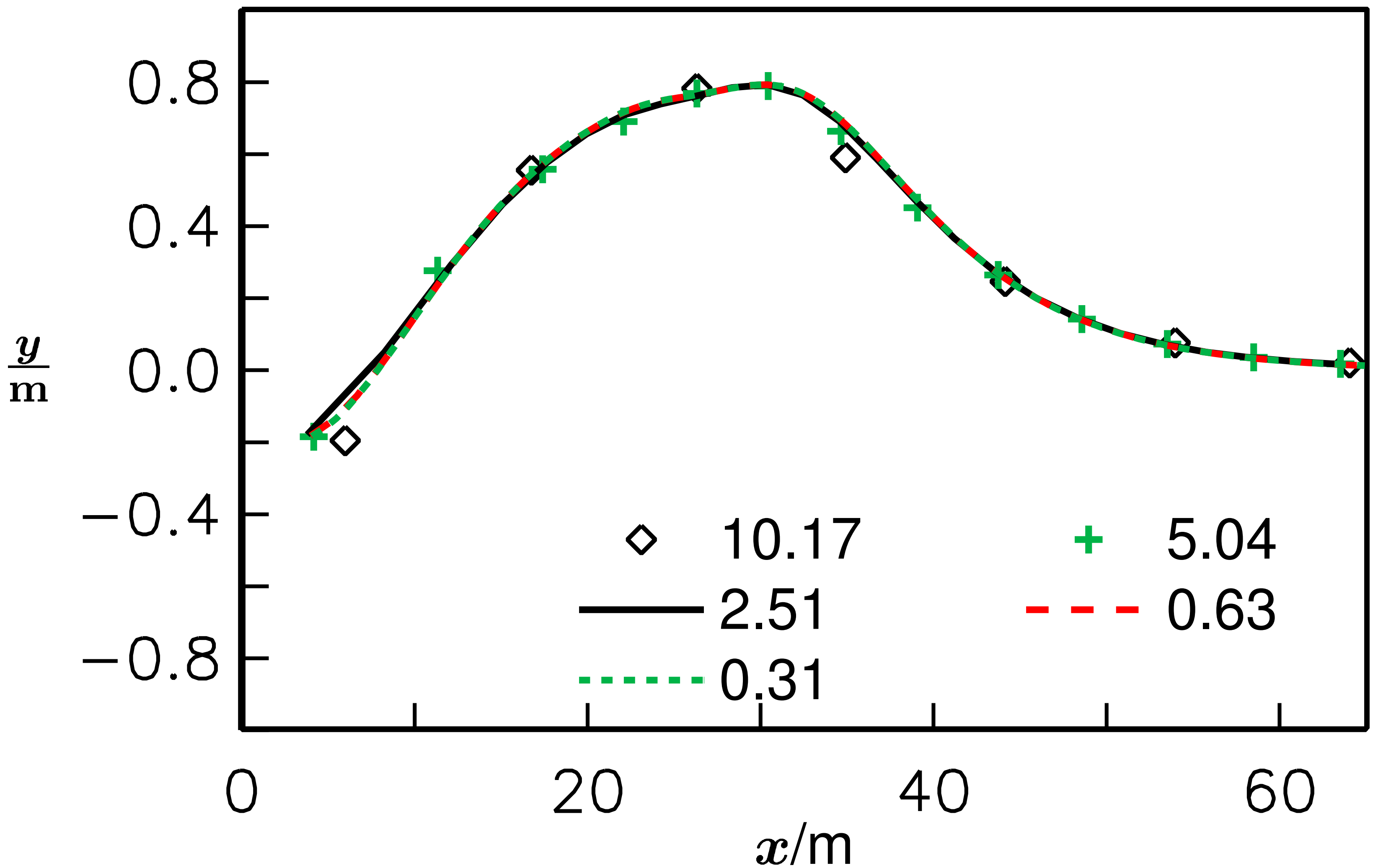
B at $t = 4.5$. Comparison of models.

A, $t = 0.5$: convergence



Bouss(NL), A: $t = 0.5$. Curves marked by Δa (m)

B, $t = 2.5$: convergence



Bouss(NL), B: $t = 2.5$. Curves marked by Δa (m)

Concluding remarks 1

Wave models

- Linear h.o. term for slide (S_1) affects wave generation slightly for case B
- Dispersion important for B when generated waves propagate into deeper water
- Even case A is affected by nonlinearity close to beach
- Linear models perform fairly well also for case B, but significant errors (save at wave front for small t).
- case A: Reference (analytic solution) superior to “naive” SW close to shore.
  (Not the case for B)

Concluding remarks 2

Performance of techniques

(Concerning the computed time span)

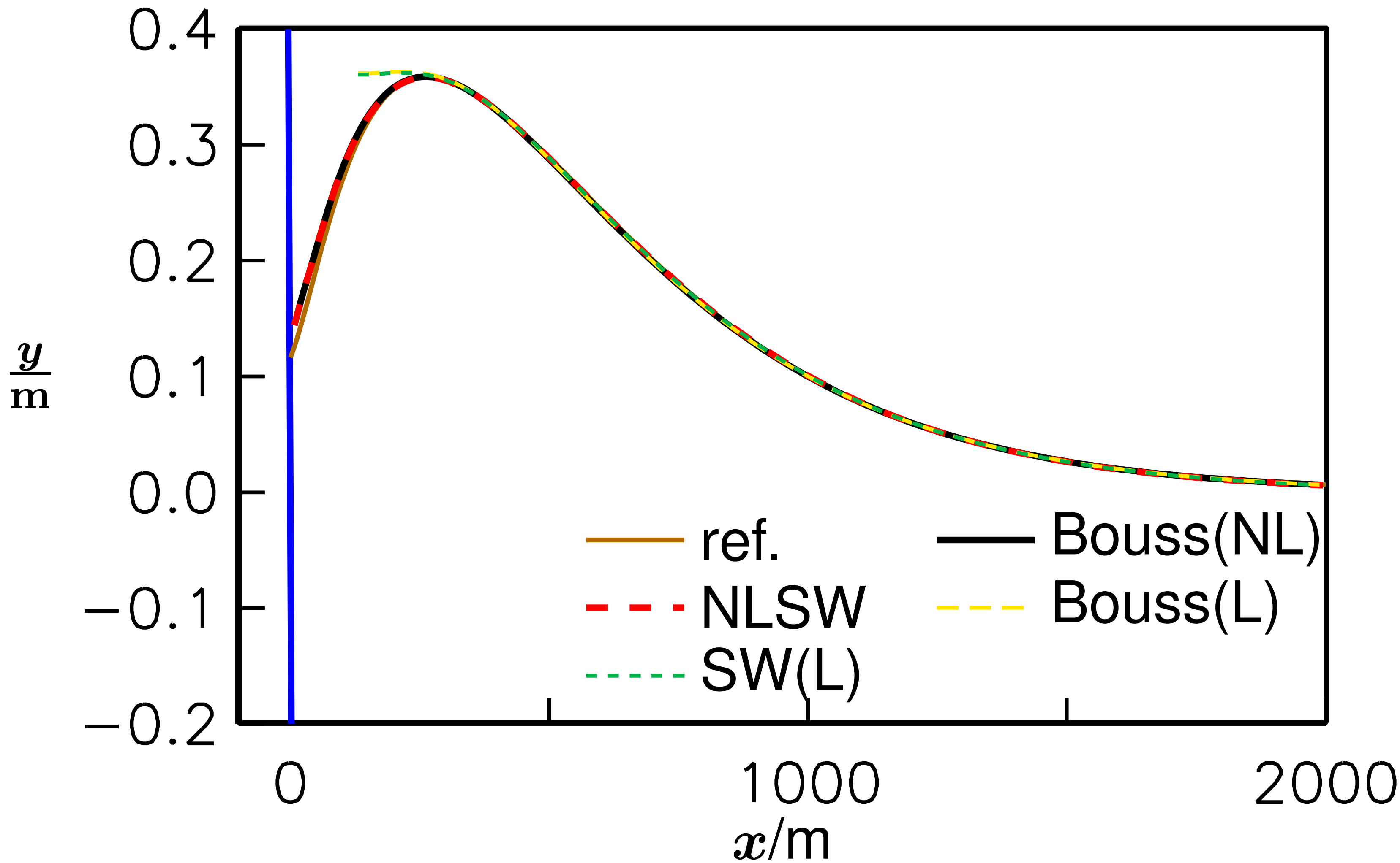
- Shoreline region most sensitive to resolution
- Case A most demanding due to large difference between length scales at beach and in deeper water.
- Less demanding than benchmark 1
- Problems in store for case A and $t > 1.5$?

Extras

The simulations

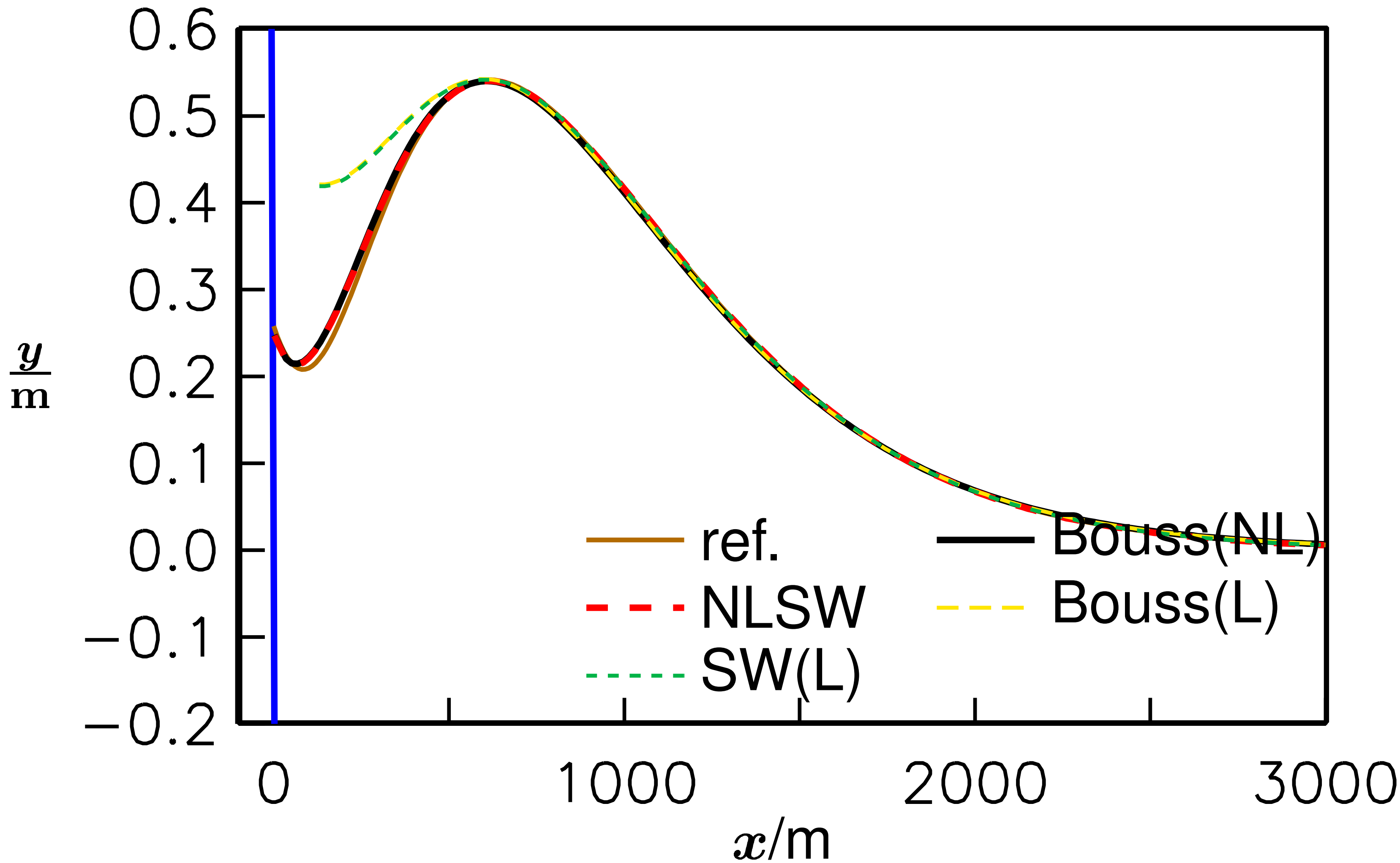
- Series of resolutions employed for each model
- Eulerian FDM: adaptive spatial refinement employed
- Lagrangian FDM: uniform resolution initially
- Different approximations in source term tested

Case A, $t = 0.5$



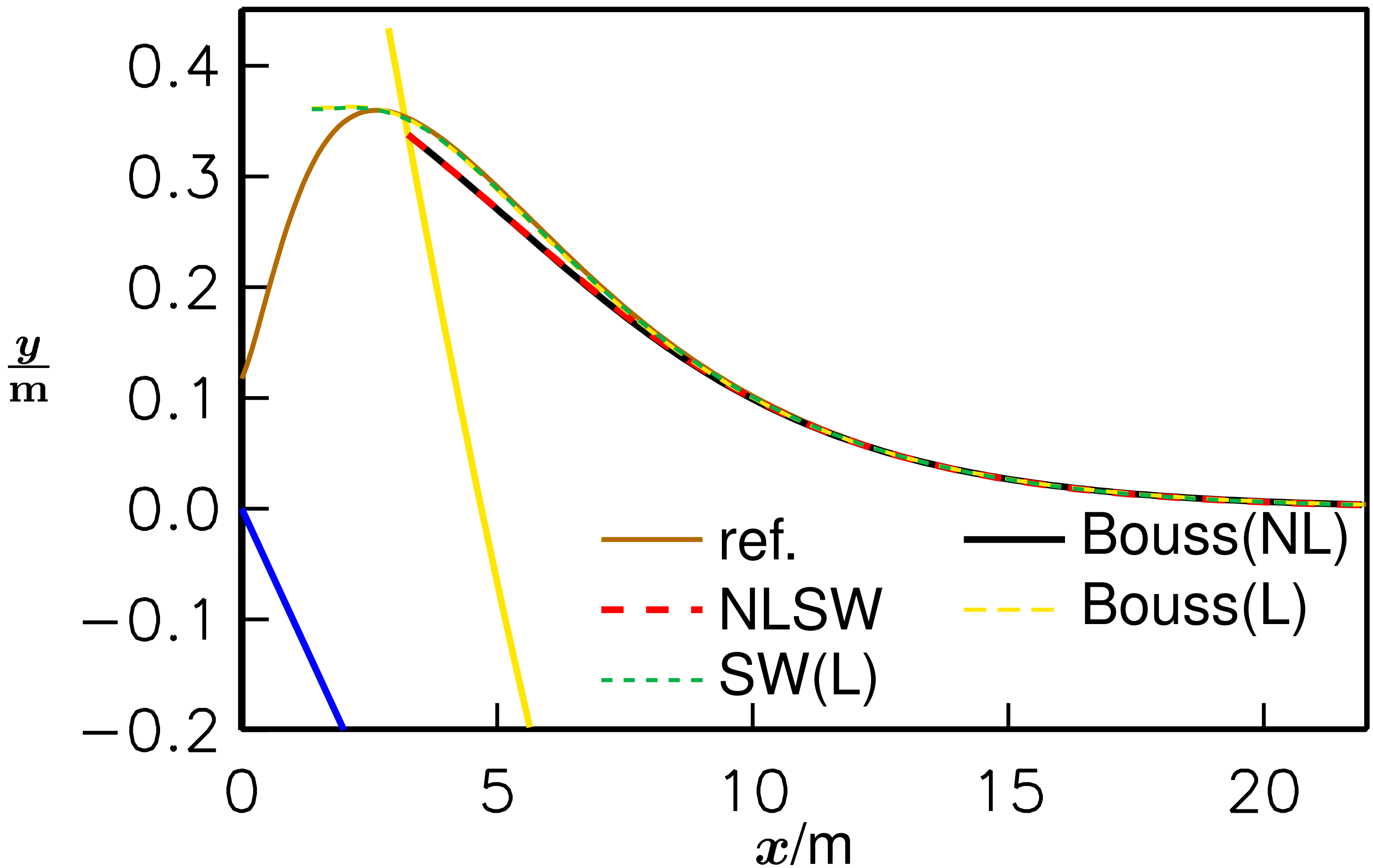
A at $t = 0.5$. Comparison of models.

Case A, $t = 1.0$



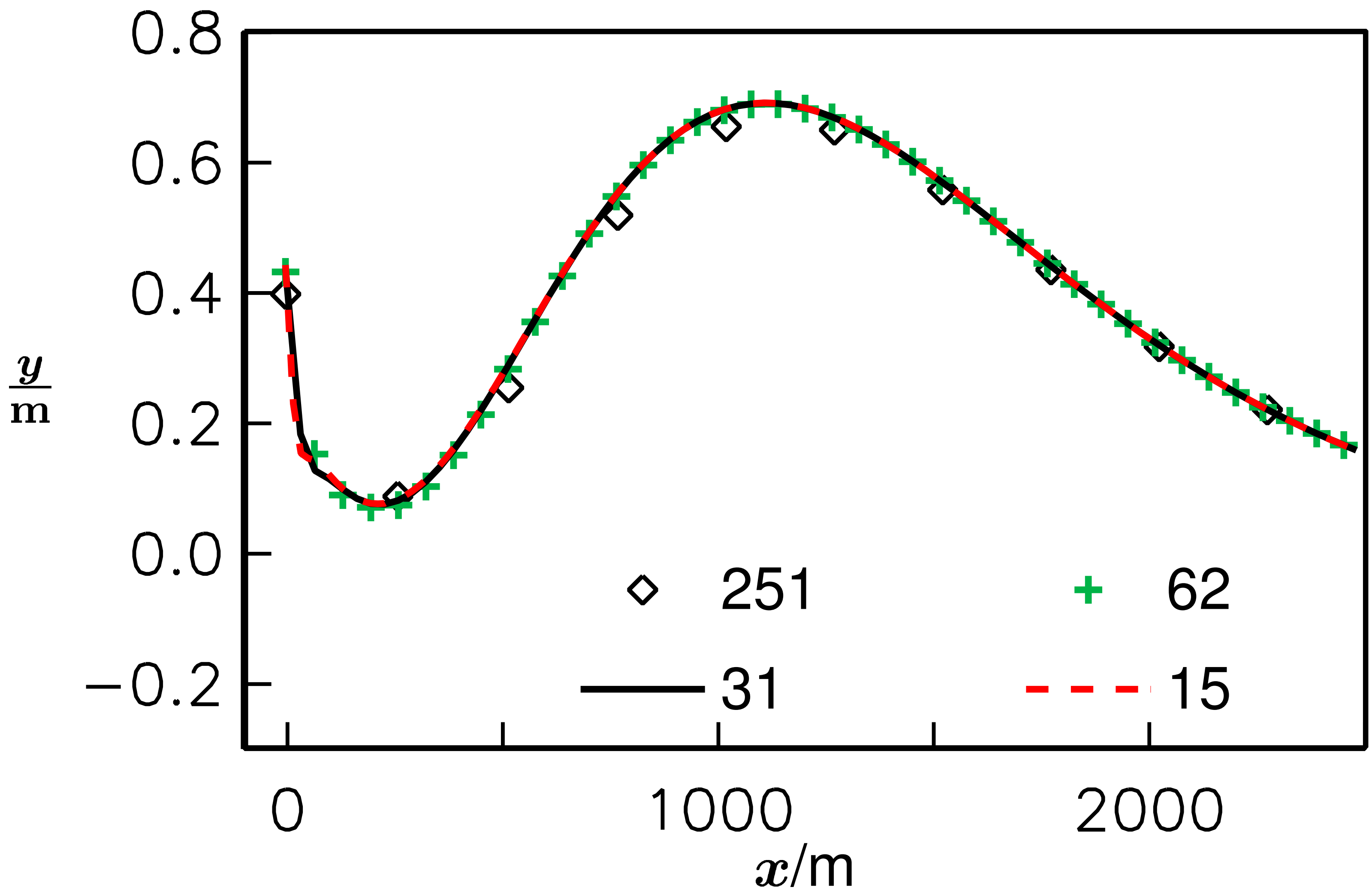
A at $t = 1.0$. Comparison of models.

Case B, $t = 0.5$



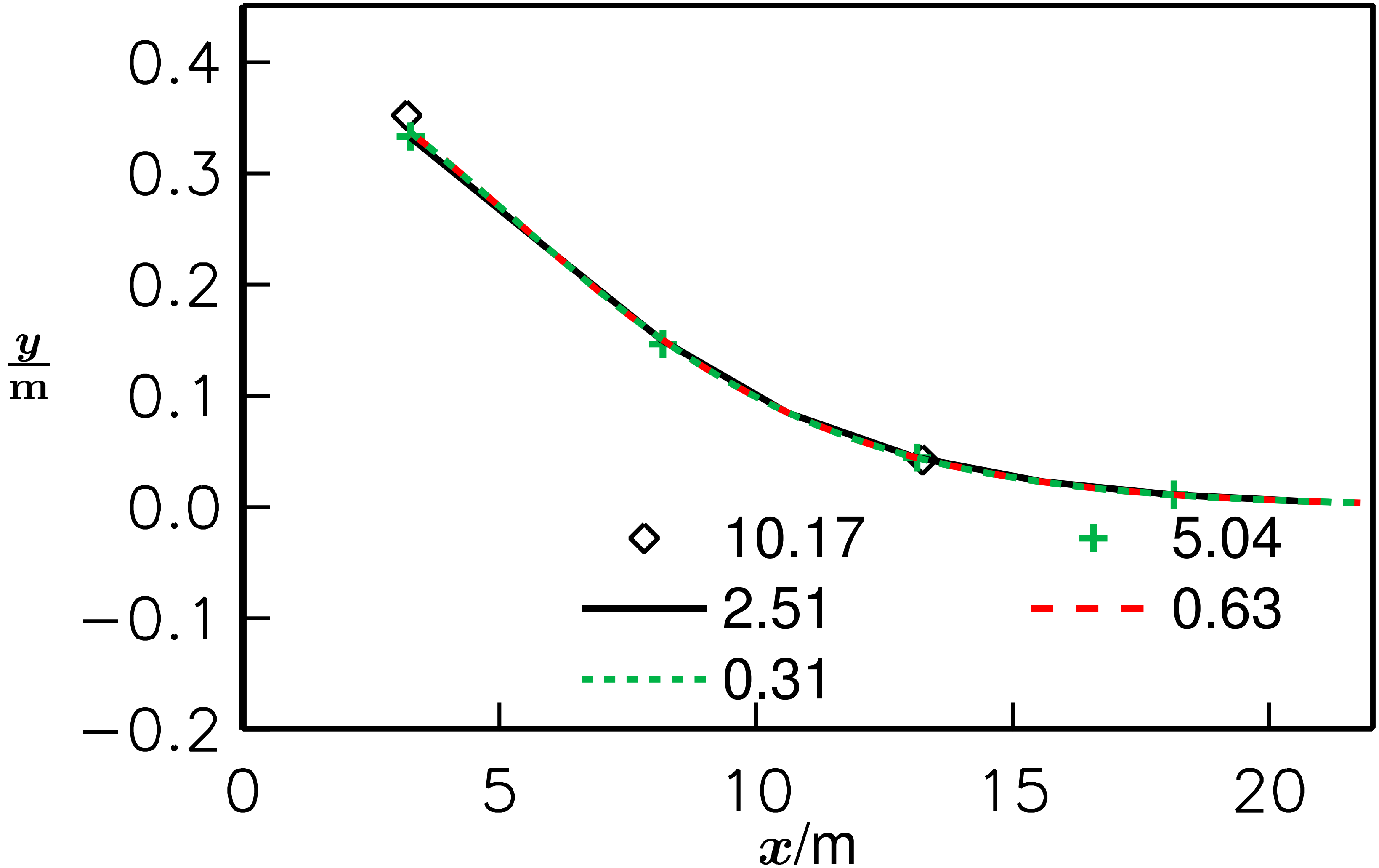
B at $t = 0.5$. Comparison of models.

A, $t = 1.5$: convergence



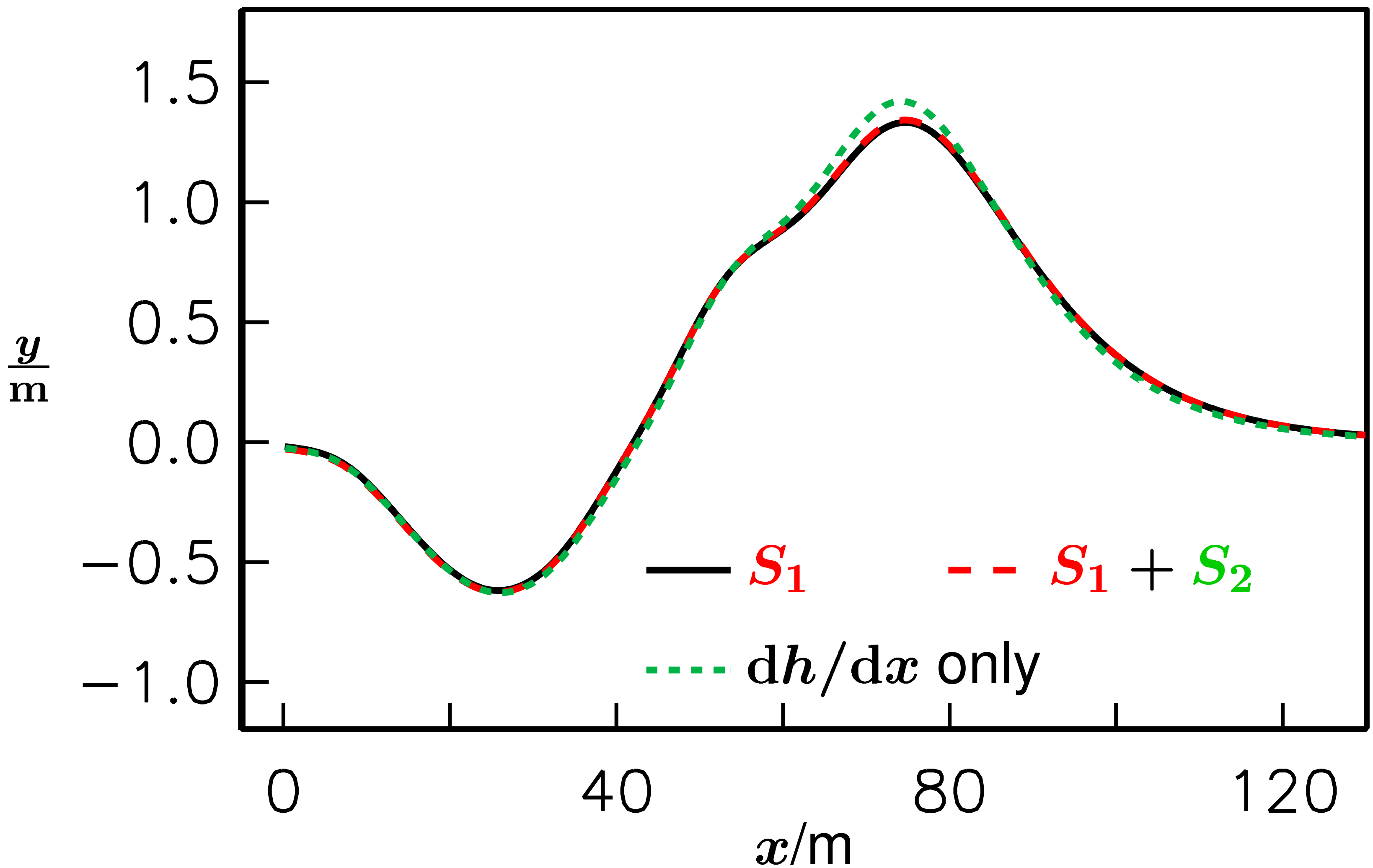
Bouss(NL), A: $t = 1.5$. Curves marked by Δa (m)

B, $t = 0.5$: convergence



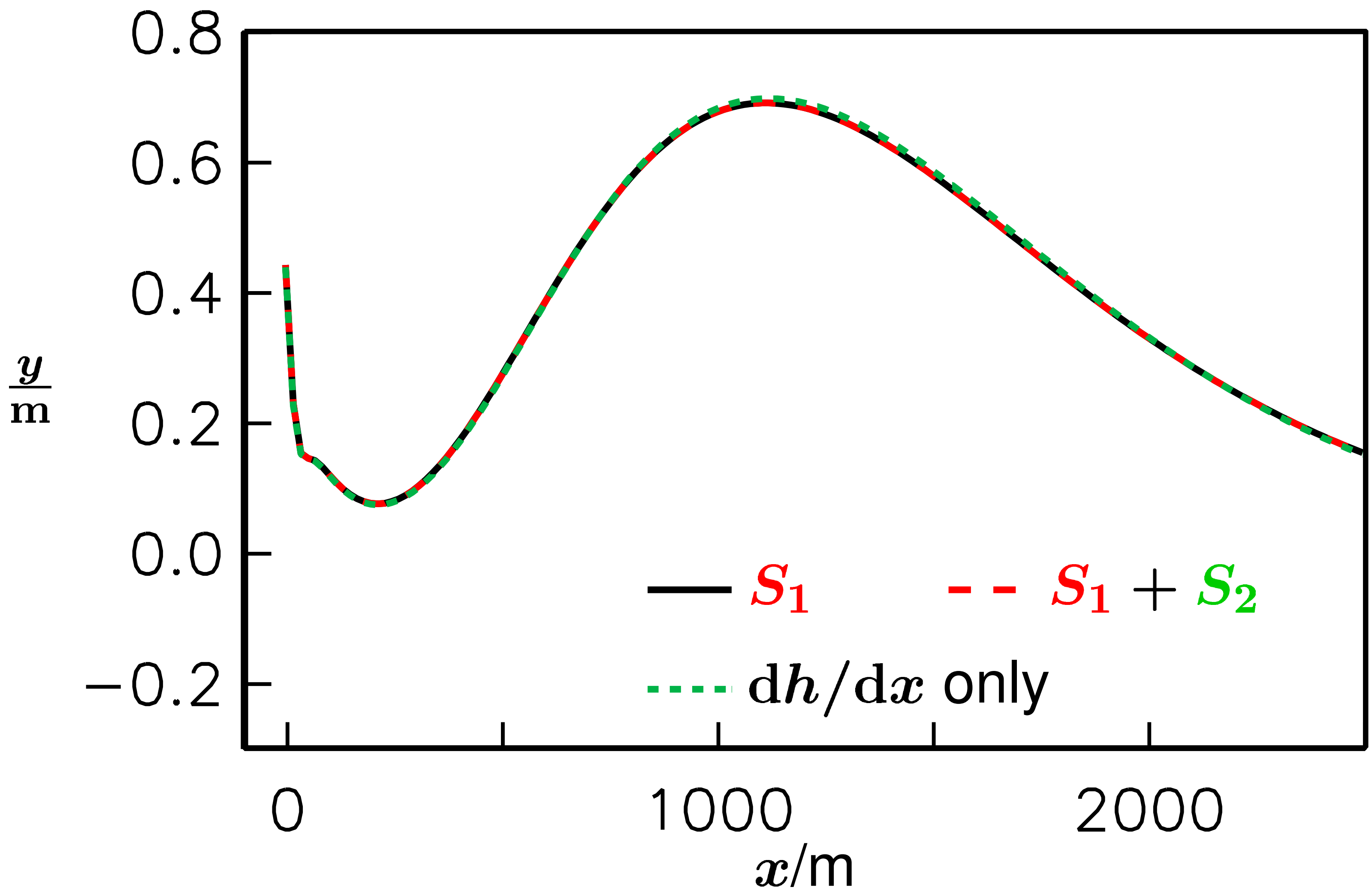
Bouss(NL), B: $t = 0.5$. Curves marked by Δa (m)

h.o. source terms



Bouss(NL), B: $t = 4.5$. Source terms

h.o. source terms



Bouss(NL), A: $t = 1.5$. Source terms

References on models and related papers

- B. Gjevik *et al.* Modeling tsunamis from earthquake sources near Gorringe Bank southwest of Portugal. *J. Geophys. Res.*, 102(C13):927–949, 1997.
- A. Jensen, G. Pedersen, and D. J. Wood. An experimental study of wave run-up at a steep beach. *J. Fluid. Mech.*, 486:161–188, 2003.
- H. Johnsgard and G. Pedersen. A numerical model for three-dimensional run-up. *Int. J. Num. Meth. Fluids*, 24:913–931, 1997.
- G. Pedersen and B. Gjevik. Run-up of solitary waves. *J. Fluid Mech.*, 135:283–299, 1983.

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- P. L.-F. Liu, P. Lynett and C. E. Synolakis Analytical solutions for forced long waves on a sloping beach. *J. Fluid Mech.*, 478:101–109, 2003.
- C. Harbitz, G. Pedersen and B. Gjevik Numerical simulation of large water waves due to landslides. *J. Hydraulic Enginering*, 119(12):1325-1342, 1993.